國立臺灣師範大學九十七學年度碩士班考試入學招生試題

線性代數 科試題（數學系用，本試題共 2 頁）

投資、投資教育 統計組

注意：1.依次序作答，只要標明題號，不必抄題。
2.答案必須寫在答案卷上，否則不予計分。

1. Let V be the vector space of all 2×3 matrices over a field F. Let W_1 be the subspace of V which consists of all matrices of the form $\begin{bmatrix} x & -x & u \\ y & z & v \end{bmatrix}$ and let W_2 be the subspace of V which consists of all matrices of the form $\begin{bmatrix} a & b & -b \\ -a & c & d \end{bmatrix}$.

(1) (7 分) Find the dimensions of W_1, W_2, and $W_1 \cap W_2$. *(You don't need to explain your answers.)*

(2) (8 分) Prove that $W_1 + W_2 = V$.

2. (8 分) Are the following two matrices row-equivalent? Explain your answer.

$$A = \begin{bmatrix} 2 & 0 & 0 \\ 4 & -1 & 0 \\ 2 & 5 & 3 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 & 1 & 2 \\ -2 & 0 & -1 \\ 1 & 3 & 5 \end{bmatrix}.$$

3. (10 分) Let T be a linear operator on a vector space V, not necessarily finite-dimensional, and suppose $T^2 = T$. Prove that $V = \ker T \oplus \operatorname{im} T$.

4. (10 分) Let A be an $n\times n$ real matrix with $A^2 = A$. Prove that A is diagonalizable.

5. (7 分) Let A be a 6×6 real matrix. Suppose the minimal polynomial for A is $p(x) = (x+1)^2(x^2+1)$. Find all the possible characteristic polynomials for A. *(You don't need to explain your answers.)*

6. Let

$$A = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 1 \end{bmatrix} \in M_{n\times n}(\mathbb{R}).$$

(1)(10 分) Find the eigenvalues and the corresponding eigenvectors of A.

(2)(5 分) Determine whether A is diagonalizable.

7. (10 分) Use Problem 6 to find the eigenvalues of the matrix $\begin{bmatrix} 1 & 1 & 1 & 1 & 2 \\ 1 & 1 & 1 & 2 & 1 \\ 1 & 1 & 2 & 1 & 1 \\ 1 & 2 & 1 & 1 & 1 \\ 2 & 1 & 1 & 1 & 1 \end{bmatrix}$.

8. (10 分) Let A be an $n\times n$ matrix. Prove that $\dim(\mathrm{span}(I, A, A^2, \ldots)) \le n$.

9. (5 分) State the definition of an inner product on a vector space V.

10. Let $\langle\cdot,\cdot\rangle$ be an inner product on $\mathbb{C}^n$.

(1)(5 分) Show that there is a unique positive definite matrix $A \in M_{n\times n}(\mathbb{C})$ such that $\langle x, y\rangle = y^* A x$ for all $x, y \in \mathbb{C}^n$.

(2)(5 分) Let $B \in M_{n\times n}(\mathbb{C})$. The linear operator L_B is defined by $L_B(x) = Bx$ for all $x \in \mathbb{C}^n$. Is the adjoint operator of L_B with respect to the inner product $\langle\cdot,\cdot\rangle$ equal to L_{B^*}? If not, find L_{B^*}.