國立臺灣師範大學九十七學年度碩士班考試入學招生試題

機率與統計 科試題（數學系用，本試題共 3 頁）

統計組

注意：1.依次序作答，只要標明題號，不必抄題。
2.答案必須寫在答案卷上，否則不予計分。

1. (20) Suppose we have an unfair coin which has a 75% chance of landing heads-up. Let X represent the difference between the number of heads and the number of tails obtained when this coin is tossed 10 times.

   (a) What is the moment-generating function of X?

   (b) What are the mean and variance of X?

2. (10) An unfair coin has a 75% chance of landing heads-up. Let $X = 1$ if it lands heads-up, and $X = 0$ if it lands tails-up. Find the sampling distribution of the **median** of samples of size 3.

3. (20) Suppose that by any time t the number of people who have gathered at the hallway outside a classroom for attending a speech is a Poisson random variable with mean λt. The doors of the classroom will be open at a time (independent of when the students arrive) that is uniformly distributed over $(0, T)$. Assuming that once the doors are open, the students would enter the classroom immediately and afterwards the doors would be shut closed again. What are the mean and variance of the number of students that enter the classroom?

4. (30) A group of 10 students will take an exam which they fail if they make 2 or more errors. The number of errors made by each student is independent of the numbers made by any other students and follows a Poisson(λ) distribution where $\lambda > 0$ is unknown.

   (a) If Harry is one of the 10 students, what is the probability that he will pass the exam?

   (b) Suppose now that 2 out of the 10 students pass the exam. What is the likelihood function of λ given this observation?

   (c) Show that the maximum likelihood estimator (MLE) of λ, say $\hat{\lambda}$, satisfies $5e^{-\hat{\lambda}}(1 + \hat{\lambda}) = 1$.

   (d) Suppose now that you had observed the number of errors made by the 10 students to be 0, 1, 1, 2, 3, 3, 4, 5, 5, 6. Give an estimate for λ from these data and state briefly your reasons for the choice of the estimator.

(e) Assume that $\hat{\lambda}$ is approximately Normally distributed with mean λ and variance λ/n, where n is the sample size. Use this result to construct an approximate 90% confidence interval for λ based on the observed values (Hint: $z_{(0.05)} = 1.64$, $z_{(0.025)} = 1.96$, $z_{(0.01)} = 2.326$).

(f) Explain briefly how this confidence interval may be used to test the null hypothesis $H_0 : \lambda = \lambda_0$ against the alternative hypothesis $H_0 : \lambda \neq \lambda_0$.

5. (20) Let $X_1, X_2, \cdots, X_n$ be a random sample of size n from a Poisson(λ) distribution. The null hypothesis $H_0 : \lambda = 1$ is to be tested against the alternative hypothesis $H_1 : \lambda = 2$. Neyman-Pearson Lemma states that the best critical region of size α for testing $H_0 : \lambda = 1$ against $H_1 : \lambda = 2$ has the form given by $R = \{(x_1, x_2, \cdots, x_n) : \sum_{i=1}^{n} x_i \geq k\}$ for some suitable k.

   (a) Calculate the value of k if $n = 5$ and $\alpha = 0.1$.

   (b) Determine the power of the resulting test in (a).

   (c) Suppose it is required that the Type I and Type II errors should each be no more than 0.01. Using the Normal approximation to the Poisson distribution (without the continuity correction), find the smallest value of n that will satisfy this requirement.

Cumulative Poisson Distribution Table

Table shows cumulative probability functions of Poisson Distribution with various α. Example: to find the probability $P(X \leq 3)$ where X has a Poisson Distribution with $\alpha = 2$, look in row 4 and column 4 to find $P(X \leq 3)$=0.8571 where X is Poisson(2).

x	α = 0.5	1	1.5	2	2.5	3	3.5	4	4.5	5
0	0.6065	0.3679	0.2231	0.1353	0.0821	0.0498	0.0302	0.0183	0.0111	0.0067
1	0.9098	0.7358	0.5578	0.4060	0.2873	0.1991	0.1359	0.0916	0.0611	0.0404
2	0.9856	0.9197	0.8088	0.6767	0.5438	0.4232	0.3208	0.2381	0.1736	0.1247
3	0.9982	0.9810	0.9344	0.8571	0.7576	0.6472	0.5366	0.4335	0.3423	0.2650
4	0.9998	0.9963	0.9814	0.9473	0.8912	0.8153	0.7254	0.6288	0.5321	0.4405
5	1.0000	0.9994	0.9955	0.9834	0.9580	0.9161	0.8576	0.7851	0.7029	0.6160
6	1.0000	0.9999	0.9991	0.9955	0.9858	0.9665	0.9347	0.8893	0.8311	0.7622
7	1.0000	1.0000	0.9998	0.9989	0.9958	0.9881	0.9733	0.9489	0.9134	0.8666
8	1.0000	1.0000	1.0000	0.9998	0.9989	0.9962	0.9901	0.9786	0.9597	0.9319
9	1.0000	1.0000	1.0000	1.0000	0.9997	0.9989	0.9967	0.9919	0.9829	0.9682
10	1.0000	1.0000	1.0000	1.0000	0.9999	0.9997	0.9990	0.9972	0.9933	0.9863
11	1.0000	1.0000	1.0000	1.0000	1.0000	0.9999	0.9997	0.9991	0.9976	0.9945
12	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9999	0.9997	0.9992	0.9980
13	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9999	0.9997	0.9993
14	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9999	0.9998
15	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9999
16	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000

x	α = 5.5	6	6.5	7	7.5	8	8.5	9	9.5	10
0	0.0041	0.0025	0.0015	0.0009	0.0006	0.0003	0.0002	0.0001	0.0001	0.0000
1	0.0266	0.0174	0.0113	0.0073	0.0047	0.0030	0.0019	0.0012	0.0008	0.0005
23	0.0884	0.0620	0.0430	0.0296	0.0203	0.0138	0.0093	0.0062	0.0042	0.0028
3	0.2017	0.1512	0.1118	0.0818	0.0591	0.0424	0.0301	0.0212	0.0149	0.0103
4	0.3575	0.2851	0.2237	0.1730	0.1321	0.0996	0.0744	0.0550	0.0403	0.0293
5	0.5289	0.4457	0.3690	0.3007	0.2414	0.1912	0.1496	0.1157	0.0885	0.0671
6	0.6860	0.6063	0.5265	0.4497	0.3782	0.3134	0.2562	0.2068	0.1649	0.1301
7	0.8095	0.7440	0.6728	0.5987	0.5246	0.4530	0.3856	0.3239	0.2687	0.2202
8	0.8944	0.8472	0.7916	0.7291	0.6620	0.5925	0.5231	0.4557	0.3918	0.3328
9	0.9462	0.9161	0.8774	0.8305	0.7764	0.7166	0.6530	0.5874	0.5218	0.4579
10	0.9747	0.9574	0.9332	0.9015	0.8622	0.8159	0.7634	0.7060	0.6453	0.5830
11	0.9890	0.9799	0.9661	0.9467	0.9208	0.8881	0.8487	0.8030	0.7520	0.6968
12	0.9955	0.9912	0.9840	0.9730	0.9573	0.9362	0.9091	0.8758	0.8364	0.7916
13	0.9983	0.9964	0.9929	0.9872	0.9784	0.9658	0.9486	0.9261	0.8981	0.8645
14	0.9994	0.9986	0.9970	0.9943	0.9897	0.9827	0.9726	0.9585	0.9400	0.9165
15	0.9998	0.9995	0.9988	0.9976	0.9954	0.9918	0.9862	0.9780	0.9665	0.9513
16	0.9999	0.9998	0.9996	0.9990	0.9980	0.9963	0.9934	0.9889	0.9823	0.9730
17	1.0000	0.9999	0.9998	0.9996	0.9992	0.9984	0.9970	0.9947	0.9911	0.9857
18	1.0000	1.0000	0.9999	0.9999	0.9997	0.9993	0.9987	0.9976	0.9957	0.9928
19	1.0000	1.0000	1.0000	1.0000	0.9999	0.9997	0.9995	0.9989	0.9980	0.9965
20	1.0000	1.0000	1.0000	1.0000	1.0000	0.9999	0.9998	0.9996	0.9991	0.9984
21	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9999	0.9998	0.9996	0.9993
22	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9999	0.9999	0.9997
23	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9999	0.9999
24	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000