Instruction:

- 20 points to each problem. Answer 5 problems from the following problems.
- No calculator, translator or other related electronic device are allowed.

1. **(20 points)** Let $f \in C^{\infty}(R)$. The n-th Taylor's expansion at $x = a$ can be expressed as

$$f(x) = p_n(x) + R_n(x)$$

where $p_n(x)$ is the n-th Taylor's polynomial and

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$$

is the remainder. Here c lies between x and a,

(a) Derive the formula for $p_n(x)$.

(b) Let $f(x) = e^x$ and $a = 0$. Find an n so that the error, $|f(x) - p_n(x)|$, is less than 5×10^{-5}. Find each coefficient of $p_n(x)$.

(c) Conclude on convergence from the table below when Taylor polynomials of different order are used to approximate e^x at $x = 0$.

x	$p_1(x)$	$p_2(x)$	$p_3(x)$	e^x
-2.0	-1.0	1.000	-0.33333	0.13534
-1.0	0.0	0.500	0.33333	0.36788
-0.5	0.5	0.625	0.60417	0.60653
-0.1	0.9	0.905	0.90483	0.90484
0.0	1.0	1.000	1.00000	1.00000
0.1	1.1	1.105	1.10517	1.10517
0.5	1.5	1.625	1.64583	1.64872
1.0	2.0	2.500	2.66667	2.71828
2.0	3.0	5.000	6.33333	7.38906

2. **(20 points)**

(a) Give an estimate for $\sqrt{2}$ by means of the Bisection Method after two steps with starting point $x_0 = 1$, and $x_1 = 2$. That is, find x_4.

(b) Give an estimate for $\sqrt{2}$ by means of the Newton's Method after two iterations with the starting point $x_0 = 1$. That is, find x_2.

(c) State the advantage and disadvantage of the Bisection Method and the Newton's Method for finding roots.

3. **(20 points)** Let $f \in C^2(R)$.

(a) Use Taylor expansion to show

$$f'(x) = \frac{f(x+h) - f(x)}{h} - \frac{f''(c)}{2}h$$

and

$$f'(x) = \frac{f(x+h) - f(x-h)}{2h} - \frac{f'''(c)}{6}h^2$$

(b) Use a computer to approximate $f'(\sqrt{2})$, where $f(x) = \tan^{-1} x$. Use

$$f'(x) \approx \frac{1}{h}[f(x+h) - f(x)]; \quad f'(x) \approx \frac{1}{2h}[f(x+h) - f(x-h)]$$

with step size h approaching 0. Recall that the exact value is $\frac{1}{3}$. Some of the output from a 32-bit machine is shown here:

$f'(x) \approx \frac{1}{h}[f(x+h) - f(x)]$		$f'(x) \approx \frac{1}{2h}[f(x+h) - f(x-h)]$	
h	value	h	value
$.62\times10^{-1}$	0.32374954	.25	0.33719385
$.24\times10^{-3}$	0.33325195	$.98\times10^{-3}$	0.33334351
$.95\times10^{-6}$	0.31250000	$.38\times10^{-5}$	0.32812500
$.15\times10^{-7}$	0.00000000	$.15\times10^{-7}$	0.00000000

i. Which approximation is better? Why?

ii. Theoretically, when h tends to 0, the numerical values should tend to the exact value. Explain why this fails in the above table.

4. **(20 points)**

$$A = \begin{pmatrix} 3 & -1 & 1 \\ 3 & 6 & 2 \\ 3 & 3 & 7 \end{pmatrix}$$

(a) Is A positive definite? Is A a diagonally dominant matrix? Prove or disprove.

(b) Find the LU factorization of A in which L is lower triangular and U is upper triangular with identity diagonal.

(c) Consider the system $Ax = b$, where $b = [1, 0, 4]^t$. Apply the Gauss-Seidel iteration to obtain $x^{(1)}$ and $x^{(2)}$ starting with $x^{(0)} = (0, 0, 0)^t$.

5. **(20 points)**

(a) Compute a divided difference table for these function values

x	-2	-1	0	1	2	3
$f(x)$	-5	1	1	1	7	25

(b) Find a polynomial of degree no greater than 5 which interpolates the above function values.

6. **(20 points)**

(a) Derive the Trapezoidal rule and the Simpson rule.

$$\int_a^b f(x)dx = \frac{b-a}{6}[f(a) + 4f(\frac{a+b}{2}) + f(b)]; \qquad \int_a^b f(x)dx = \frac{b-a}{2}[f(a) + f(b)].$$

(b) Apply the Simpson rule and the Trapezoidal rule separately to calculate $\int_0^1 x^3 + 2x^2 + 1dx$ with 4 equal subintervals.