國立臺灣師範大學 100 學年度碩士班招生考試試題

科目：**線性代數與代數**　　　　適用系所：**數學系**

注意：1.本試題共 1 頁，請依序在答案卷上作答，並標明題號，不必抄題。2.答案必須寫在指定作答區內，否則不予計分。

1. Let G be a group of order 168.
   (1) Does G have a subgroup of order 5? Please explain your answer. (5 分)
   (2) Does G have a subgroup of order 3? Please explain your answer. (5 分)
   (3) Suppose G is a simple group. How many Sylow 7-subgroups does G have? Please explain your answer. (7 分)
2. Suppose $G=\langle a\rangle$ is a cyclic group of order 20. Let $H=\langle a^8\rangle$ be the subgroup of G generated by a^8.
   (1) What is the order of H? You do not need to explain your answer. (5 分)
   (2) Prove that the groups G/H and Z_4 are isomorphic. (7 分)
3. Consider the polynomial $f(x)=x^3-30x^2+18x-12$ in $Q[x]$. Is $f(x)$ irreducible in $Q[x]$? Please explain your answer. (7 分)
4. Consider the polynomial $g(x)=x^3+2x+2$ in $Z_5[x]$. Let $I=\langle g(x)\rangle$ be the ideal of $Z_5[x]$ generated by $g(x)$. Is $Z_5[x]/I$ a field? Please explain your answer. (7 分)
5. Write 220 as a product of irreducible elements in $Z[i]$. You do not need to explain your answer. (7 分)
6. Let A be an 5×5 symmetric matrix such that the null space of A is
$$N(A)=\{(x_1,x_2,x_3,x_4,x_5): x_1-7x_2+5x_3-4x_4=0\}.$$
   Find a basis of the column space of A. Please explain your answer. (10 分)
7. Let A be an $m\times n$ matrix. Prove that if P and Q are invertible $m\times m$ and $n\times n$ matrices, respectively, then $rank(PAQ)=rank(A)$. (10 分)
8. Let $A=\begin{pmatrix}1&1&-1\\-1&0&1\\1&2&-1\\1&1&0\end{pmatrix}$ and $b=\begin{pmatrix}6\\-2\\10\\3\end{pmatrix}$.
   (1) Determine whether the linear system $Ax=b$ is consistent or not. (4 分)
   (2) Find the least square solution of $Ax=b$. (7 分)
9. Let $A=\begin{pmatrix}3&2&-1\\1&4&-1\\1&2&1\end{pmatrix}$. Test A for diagonalizability. (7 分)
10. Let T and U be two linear transformations on a finite-dimensional vector space V.
    (1) Show that if $[T]_\beta=[U]_\beta$ for some ordered basis β for V, then $T=U$. (6 分)
    (2) Prove that if T and U are simultaneously diagonalizable, then T and U commute (i.e., $TU=UT$). (6 分)