國立臺灣師範大學 100 學年度碩士班招生考試試題

科目：**基礎數學**　　　　適用系所：**數學系**

注意：1.本試題共 2 頁，請依序在答案卷上作答，並標明題號，不必抄題。2.答案必須寫在指定作答區內，否則不予計分。

Let $\mathbb{R}$ be the field of real numbers.

1. Let $V = \mathrm{M}_2(\mathbb{R})$ be the vector space of all 2×2 matrices over $\mathbb{R}$ and let

$$A = \begin{pmatrix} 3 & 1 \\ 6 & 2 \end{pmatrix}.$$

Suppose that $T : V \to V$ is the linear operator on V defined by $T(C) = AC$ for any $C \in V$. Moreover, let $\mathbf{B} = \{v_1, v_2, v_3, v_4\}$ where

$$v_1 = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, v_2 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, v_3 = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, v_4 = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}.$$

(a) [6 points] Show that $\mathbf{B}$ is a basis for V.

(b) [6 points] Find the null space of T.

(c) [6 points] Find the rank of T.

(d) [8 points] Find the matrix representation $[T]_{\mathbf{B}}$ of T in the ordered basis $\mathbf{B}$.

(e) [8 points] Find the characteristic polynomial of T.

2. Consider the matrix

$$A = \begin{pmatrix} 6 & -6 & -4 \\ 4 & -4 & -4 \\ 0 & 0 & 2 \end{pmatrix} \in \mathrm{M}_3(\mathbb{R}).$$

(a) [8 points] Find all the eigenvalues of A.

(b) [8 points] Determine if A is diagonalizable.

3. Evaluate the following statements:

(a) [4 points] $\int \sec^3 x \, dx$

(b) [4 points] $\int_{-1}^{1} \frac{1}{x^2} \, dx$

(c) [4 points] $\lim_{x \to 0^+} (\sin x)^x$

(d) [4 points] $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2^n n}$

(e) [4 points] $\int_2^4 \frac{\sqrt{\ln(9-x)}\, dx}{\sqrt{\ln(9-x)} + \sqrt{\ln(x+3)}}$.

(f) [4 points] $g(x) = \sin(x^3)$. Find $g^{(9)}(0)$.

(g) [4 points] $f(x, y) = \int_x^y \sqrt{1+t^3}\, dt$. Find $f_x(x, y)$.

4. The region **A** is common to the two regions bounded by the curves $r = -6\cos\theta$ and $r = 2 - 2\cos\theta$.

   (a) [5 points] Find the area of **A**.

   (b) [5 points] Find the arc length of the circumference of **A**.

   (c) [5 points] Find the area of the surface formed by revolving **A** about the line $\theta = \pi/2$.

5. [7 points] Suppose $y = f(u)$ is a differentiable function of u and $u = g(x)$ is a differentiable function of x. Prove that $y = f(g(x))$ is a differentiable function of x and $\frac{d}{dx}[f(g(x))] = f'(g(x))g'(x)$.