招生學年度	100	招生類別	碩士班
系所班別	應用數學系 統計碩士班		
科目	基礎數學		
注意事項	本考科禁止使用掌上型計算機；含微積分及線性代數		

1. (10%) Consider the following two functions defined on $\mathbb{R}^2$.

$$g(\boldsymbol{x}) = x_1^2 - x_2^2, \qquad f(\boldsymbol{x}) = x_1^2 + x_2^2 + \cos(x_1 - x_2),$$

where $\boldsymbol{x} = (x_1, x_2)$. Show that

$$\lim_{\|\boldsymbol{x}\|\to\infty} f(\boldsymbol{x}) = \infty \text{ but } \lim_{\|\boldsymbol{x}\|\to\infty} g(\boldsymbol{x}) \text{ does not exist,}$$

where $\|\boldsymbol{x}\| = \sqrt{x_1^2 + x_2^2}$.

2. (10%) Let f be a real-valued continuous function defined on an open interval $I \subset \mathbb{R}$ such that $\forall\, y \in I$ the set

$$\{x \in I : f(x) \le f(y)\} \text{ is a finite union of disjoint closed intervals.}$$

Show that f is bounded below and attains its minimum.

3. Let $g(y) = -\frac{y^2}{2} + |y|$ be a function defined on $\mathbb{R}$.
   (a) (5%) Find the definite integral of g over the interval $[-1, 1]$.
   (b) (10%) Analyze and sketch the graph of the function g, including the following issues: monotonicity, concavity, and relative extrema.

4. (20%) Let $f : \mathbb{R}^3 \to \mathbb{R}$ be defined by $f(x, y, z) = x^3 + xyz + y^2 - 3x$. Find all the stationary points, and determine whether they correspond to local extrema or saddle points based on the analysis of eigenvalues of the Hessian matrix of f.

5. (10%) Let A be an $m \times n$ real matrix of rank m, and B an $n \times r$ real matrix of rank n. Show that $\operatorname{rank}(AB) = m$.

6. (15%) Consider a vector-valued function η of three variables $\boldsymbol{\theta} = (\mu_0, \mu_1, \epsilon)'$ that is defined by

$$\eta(\boldsymbol{\theta}) = \left(\mu_0,\ \mu_1,\ \log\frac{\epsilon}{1-\epsilon} - \frac{1}{2}(\mu_1^2 - \mu_0^2)\right)',$$

where $\mu_0, \mu_1 \in \mathbb{R}$, and $\epsilon \in (0, 1)$. Find the 3×3 matrix $\frac{\partial \eta'}{\partial \boldsymbol{\theta}}$, where

$$\frac{\partial \eta'}{\partial \boldsymbol{\theta}} \equiv \begin{pmatrix} \frac{\partial \eta'}{\partial \mu_0} \\ \frac{\partial \eta'}{\partial \mu_1} \\ \frac{\partial \eta'}{\partial \epsilon} \end{pmatrix}.$$

Also determine if $\frac{\partial \eta'}{\partial \boldsymbol{\theta}}$ is nonsingular for all $\boldsymbol{\theta}$ in $\mathbb{R} \times \mathbb{R} \times (0, 1)$.

7. Let A be an $n \times n$ positive definite matrix, V be an $m \times n$ matrix of rank $m < n$, and $P = I - A^{-1}V^t(VA^{-1}V^t)^{-1}V$.
   (a) (5%) Show that the matrix $VA^{-1}V^t$ is positive definite and hence is nonsingular.
   (b) (10%) Show that $P^2 = P$ and $(I - P)^t AP = O$.
   (c) (5%) Define an inner product on $\mathbb{R}^n$ so that P can be viewed as an orthogonal projection matrix onto its range space with respect to this inner product.