1. Determine constants, a, b,…, f, so that the function $y_0(t) = a$, $y_1(t) = b + c\,t$, $y_2(t) = d + e\,t + f\,t^2$ form an orthonormal set on the interval $0 \le t \le 1$. (20%)

2. Using the Laplace transformation, solve the initial value problem

$$\frac{dt_1}{dx} = 2t_1 - 3t_2$$
$$\frac{dt_2}{dx} = t_2 - 2t_1$$
$$t_1(0) = 8, t_2(0) = 3$$

What are the characteristic equation and the eigenvalues of the problem? Explain whether the problem is stable or not. (15%)

3. The microbial growth is described by the logistic equation

$$\frac{dx(t)}{dt} = \mu_m \left(1 - \frac{x(t)}{x_m}\right) x(t)$$

where μ_m is the maximum specific growth rate (h^{-1}) and x_m is the maximum attainable biomass concentration (g/L). Both factors are constants. Find the biomass concentration $x(t)$ as a function of time t if the initial biomass concentration is set as x_0. What is the biomass concentration when the time approaches to infinity? (15%)

4. The method of Gauss-Jordan elimination is to get an inverse of a square matrix **A** from the augmented matrix [**A** | **I**] to [**I** | **K**]. Prove **K** is the inverse of matrix **A**. (10%)

5. Suppose x, y, z are functions of variables q_1, q_2, q_3. Let $\bar{u}_i$ be a unit vector in the direction of increasing q_i. In any orthogonal coordinates,

(i) prove that the divergence of $\bar{F}$ can be expressed as

$$\nabla \bullet \bar{F}(q_1, q_2, q_3) = \frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial q_1}(F_1 h_2 h_3) + \frac{\partial}{\partial q_2}(F_2 h_1 h_3) + \frac{\partial}{\partial q_3}(F_3 h_1 h_2)\right],$$

where $\bar{F}(q_1, q_2, q_3) = F_1(q_1, q_2, q_3)\bar{u}_1 + F_2(q_1, q_2, q_3)\bar{u}_2 + F_3(q_1, q_2, q_3)\bar{u}_3$

$$h_1=\sqrt{(\frac{\partial x}{\partial q_1})^2+(\frac{\partial y}{\partial q_1})^2+(\frac{\partial z}{\partial q_1})^2}, h_2=\sqrt{(\frac{\partial x}{\partial q_2})^2+(\frac{\partial y}{\partial q_2})^2+(\frac{\partial z}{\partial q_2})^2},$$

$$h_3=\sqrt{(\frac{\partial x}{\partial q_3})^2+(\frac{\partial y}{\partial q_3})^2+(\frac{\partial z}{\partial q_3})^2}$$ (10%)

(ii) Show the divergence becomes $\nabla \bullet \vec{F}(r,\theta,z)=\frac{1}{r}\frac{\partial}{\partial r}(rF_r)+\frac{1}{r}\frac{\partial F_\theta}{\partial \theta}+\frac{\partial F_Z}{\partial z}$ in cylindrical coordinates. (5%) (Hint: $x = r\cos\theta, y = r\sin\theta$, z = z)

6. The vibration of an elastic string of length L, fastened at the ends, picked up at time zero to assume the configuration of the graph of $y = g(x)$, and released from rest.

(i) Prove the governing equation is the wave function,

$\frac{\partial^2 y}{\partial t^2}=c^2\frac{\partial^2 y}{\partial x^2}$, $0< x < L$, t >0, where c^2 is a constant. (10%)

(ii) Solve the differential equation. (15%)