1. **(10%)** Show that the matrix $A$ is invertible for all values of $\theta$, and find $A^{-1}$.

$$A = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & -2 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}$$

2. (a) **(5%)** Based on standard matrix addition and scalar multiplication, determine whether the set consisting of $U$, $V$, $W$ is linearly independent or not.

$$U = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, V = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, W = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

(b) **(5%)** Find a matrix $X$ with only one nonzero entry such that the set $\{U, V, W, X\}$ forms a basis for the vector space including all $2 \times 2$ matrices, and explain why.

(c) **(5%)** Let $A = U + V + W$, express $A$ as a product of elementary matrices.

3. **(10%)** Let $A = \begin{bmatrix} 2 & 1 \\ 4 & -1 \end{bmatrix}$. Write $A = PDP^{-1}$, where $D$ is a diagonal matrix.

4. **(10%)** Let $W = span\{v_1, v_2\}$ be a subspace of $R^4$, $v_1$ and $v_2$ are orthogonal with respect to the Euclidean inner product where $v_1 = (0, 1, 4, -1)$ and $v_2 = (3, 5, 1, 1)$. Given a vector $x = (1, 2, 0, -2)$, find the vectors $w_1$ in $W$ and $w_2$ in $W^{\perp}$ such that $x = w_1 + w_2$, where $W^{\perp}$ is the orthogonal complement of $W$ in $R^4$.

5. **(5%)** Let $S = \{v_1, v_2, v_3\}$ be a basis for $R^3$, where $v_1 = (1, 1, 1)$, $v_2 = (1, 1, 0)$, and $v_3 = (1, 0, 0)$. Let $T : R^3 \to R^3$ be the linear operator such that $T(v_1) = (-1, 2, 4)$, $T(v_2) = (0, 3, 2)$, and $T(v_3) = (1, -5, -1)$. Find a formula for $T(x_1, x_2, x_3)$.

6. **(10%)** A set of propositions is consistent if there is an assignment of truth values to each of the variables in the propositions that makes each proposition true. Is the following set of propositions consistent? Prove your answer.

The system is in multiuser state if and only if it is operating normally.

If the system is operating normally, the kernel is functioning.

The kernel is not functioning or the system is in interrupt mode.

If the system is not in multiuser state, then it is in interrupt mode.

The system is in interrupt mode.

7. **(7%)** Suppose that the only currency were 3-dollar bills and 10-dollar bills. Show that any dollar amount greater than 17 dollars could be made from a combination of these bills.

8. **(8%)** Consider the recurrence relation $a_n = 2a_{n-1} + 3n$.

(a) (1%) Write the associated homogeneous recurrence relation.

(b) (2%) Find the general solution to the associated homogeneous recurrence relation.

(c) (2%) Find a particular solution to the given recurrence relation.

(d) (1%) Write the general solution to the given recurrence relation.

(e) (2%) Find the particular solution to the given recurrence relation when $a_0 = 1$.

9. **(10%)** If $a$ and $b$ are integers and $m$ is a positive integer, then $a$ is congruent to $b$ modulo $m$ if $m$ divides $a - b$. We use the notation $a \equiv b \pmod{m}$ to indicate that $a$ is congruent to $b$ modulo $m$.

(a) **(5%)** Find an inverse of 5 modulo 47.

(b) **(5%)** Solve the congruence $5x \equiv 7 \pmod{47}$

10. **(10%)** How many numbers must be selected from the first 10 positive integers to guarantee that at least two pairs of these numbers add up to 10?

11. **(5%)** Let $G$ be the grammar with vocabulary $V = \{S, A, B, a, b\}$, set of terminals $T = \{a, b\}$, starting symbol $S$, and productions $P = \{S \to AB, A \to aAb, B \to bBa, A \to ab, B \to ba\}$. What is $L(G)$, the language generated by $G$?