系所組別： 數學系應用數學

考試科目： 線性代數

考試日期：0219，節次：2

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(1) (25 %) Please do the following problems.

(a) (8 %) State the definition of a vector space $(V, +, \cdot, F)$. Are $(\mathbb{R}^n, +, \cdot, \mathbb{C})$ and $(\mathbb{C}^n, +, \cdot, \mathbb{R})$ vector spaces? State your reason.

(b) (5 %) If W and S are subspaces of V, are $W \cup S$, $W \cap S$ and $W + S$ subspaces? State your reason..

(c) (6 %) Let $\{x_1, x_2, \cdots, x_k\}$ be a linearly independent subset of a vector space V, prove that $\{x_1 - x_k, x_2 - x_k, \cdots, x_{k-1} - x_k, x_k\}$ is also linearly independent.

(d) (6 %) Let $W = \{\begin{bmatrix} a & c \\ c & b \end{bmatrix} \mid a, b, c \in \mathbb{R}\}$. Show that W is a subspace of $M_{2\times 2}(\mathbb{R})$, where $M_{2\times 2}(\mathbb{R})$ is the set of 2×2 matrices with entries in $\mathbb{R}$, and find $\dim W$.

(2) (30 %) Let $T : V \to W$ be linear, where V an W are vector spaces over the same field F.

(a) (10 %) T is said to be independence preserving if $T(I)$ is linearly independent in W whenever I is linearly independent in V. Prove that T is independence preserving if and only if T is one-to-one.

(b) (20 %) If $V = W$ and $\dim V < +\infty$, show that

(i) (5 %) λ is an eigenvalue of T if and only if $p(\lambda) = 0$, where p is the minimal polynomial of T.

(ii) (5 %) Let $V = W = \mathbb{R}^3$ and $T(x, y, z) = (-x + y - z, y, 3x - y + 3z)$. Find the minimal polynomial of T.

(iii) (10 %) Let M be a non-zero and proper subspace of $\mathbb{R}^3$ and P be the orthogonal projection from $\mathbb{R}^3$ onto M. Find the matrix representation of P with respect to the standard basis.

(3) (15 %) Please do the following problems.

(a) (6 %) Find new coordinates x', y' so that the following quadratic form can be written as $\lambda_1(x')^2 + \lambda_2(y')^2$.

$$3x^2 + 2xy + 3y^2$$

(b) (9 %) Let $A = \begin{bmatrix} 4 & 2 & 2 \\ 2 & 4 & 2 \\ 2 & 2 & 4 \end{bmatrix}$. Find an orthogonal matrix P and a diagonal matrix D such that $P^{-1}AP = D$.

(4) (20 %) Let $\mathbf{P}_2(\mathbb{R}) = \{ax^2 + bx + c \mid a, b, c \in \mathbb{R}\}$ and define the inner product $\langle \cdot, \cdot \rangle$ by

$$\langle f, g \rangle = \int_0^1 f(t)g(t)\,dt, \quad \forall f, g \in \mathbf{P}_2(\mathbb{R}).$$

Let T be defined by $T(f) = f' + 3f$, $\forall f \in \mathbf{P}_2(\mathbb{R})$.

(背面仍有題目,請繼續作答)

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(a) (4 %) Show that T is linear and find the matrix representation of P with respect to the basis $\mathcal{B} = \{1, x, x^2\}$, denoted by $[T]_{\mathcal{B}}$.

(b) (6 %) Is $[T]_{\mathcal{B}}$ diagonalizable? If not, find a matrix Q such that $Q^{-1}[T]_{\mathcal{B}}Q$ is the Jordan form of $[T]_{\mathcal{B}}$.

(c) (10 %) Find $T^*(f)$,where $f(x) = 6x^2 - 4x + 1$.

(5) (10 %) Let T and U be linear operators on $\mathbb{R}^3$ defined by

$$T(x, y, z) = (-3x + 3y - 2z, -7x + 6y - 3z, x - y + 2z)$$

and

$$U(x, y, z) = (y - z, -4x + 4y - 2z, -2x + y + z), \quad \forall (x, y, z) \in \mathbb{R}^3.$$

Show that T and U are not similar by finding their Jordan canonical forms.