系所組別： 電腦與通信工程研究所乙組
考試科目： 通信數學

※ 考生請注意：本試題 □可 ☑不可 使用計算機

1. (10%) Among 33 students in a class, 17 of them earned A's on the midterm exam, 14 earned A's on the final exam, and 11 did not earn A's on either examination. What is the probability that a randomly selected student from this class earned an A on both exams?

2. (15%) Suppose that three numbers are selected one by one, at random and without replacement from the set of numbers $\{1, 2, 3, \ldots, n\}$. What is the probability that the third number falls between the first two if the first number is smaller than the second?

3. (15%) Prove that if X is a positive, continuous, memoryless random variable with distribution function F, then $F(t) = 1 - e^{-\lambda t}$, for some $\lambda > 0$. This shows that the exponential is the only distribution on $(0, \infty)$ with the memoryless property.

4. (10%) Let X_1, X_2, X_3, and X_4 be four independently selected random numbers from $(0, 1)$. Find $P(1/4 < X_{(3)} < 1/2)$. $X_{(3)}$ is the the 3rd smallest value in $\{X_1, X_2, X_3, X_4\}$.

5. (25%) Mark each of the following statements True (T) or False (F). (Need NOT to give reasons.)

   (a) A real square matrix may have complex eigenvalues and complex eigenvectors.

   (b) Let M be a symmetric matrix. If M is invertible, then M^{-1} is also a symmetric matrix.

   (c) Let M be a real square matrix of size n. If $\|M\mathrm{x}\|^2 = \|\mathrm{x}\|^2$ for all $\mathrm{x} \in \mathbb{R}^n$, then M is an orthogonal matrix, $M^T M = I_n$.

   (d) Let M be an $m \times n$ matrix, $m \neq n$. We have $\mathrm{rank}(M^T M) = \mathrm{rank}(MM^T)$.

   (e) Let M be an $m \times n$ matrix, $m \neq n$. We have $\mathrm{nullity}(M^T M) = \mathrm{nullity}(MM^T)$.

6. (15%) Suppose that A is a square matrix of size n, and $\lambda_1, \ldots, \lambda_k$ are distinct eigenvalues of A, with the corresponding multiplicity $m_1, \ldots, m_k$, respectively, where $m_1 + \cdots + m_k = n$. Prove the determinant of A is

$$\det(A) = \lambda_1^{m_1} \lambda_2^{m_2} \cdots \lambda_k^{m_k}.$$

7. (10%) Let I_m and I_n be identity matrices of sizes m and n, respectively, where we assume $m > n$. Can you find an $m \times n$ matrix A and an $n \times m$ matrix B such that $AB = I_m$ and $BA = I_n$? (Explain your answer.)