Part I: Linear Algebra [50 points]

Multiple Choice (1-8: three points each): each question has <u>exactly one</u> correct answer.

1. How many of these statements are true? A,B,C are matrices.
- If the products AB and BC exist, then AC exists.
- If $AB = AC$, then $B = C$
- $(A - B)(A + B) = A^2 - B^2$
- If A is invertible and $AB = 0$, then $B = 0$.
- A and B are square matrices of the same size. If AB is invertible, then A and B are also invertible.
- If A is a 3×5 matrix, then the largest possible rank of A is 5.

(a) 2 (b) 3 (c) 4 (d) 5 (e) 6

2. How many of these statements are true?
- A set of vectors that spans $\mathbf{R}^2$ is always linearly independent.
- A set of vectors of $\mathbf{R}^3$ that is linearly independent always spans $\mathbf{R}^3$.
- The rank of A and the rank of A^t are always the same. A^t is the transpose of A.
- For any square matrix A, $A + A^t$ is symmetric.
- For any square matrix A, AA^t is symmetric.
- If $|A - B| = 0$ then $|A| = |B|$

(a) 2 (b) 3 (c) 4 (d) 5 (e) 6

3. Consider the sets of vectors (a,b,c) of the following form. Determine how many of these sets form the subspaces of $\mathbf{R}^3$.

$(a, b, 0)$;　　　　$(a, b, 2)$;　　　　$(a, b, 2a + 3b)$;　　　$a + 2b - c = 0$;　　　$ab = 0$;　　　$ab = ac$

(a) 2 (b) 3 (c) 4 (d) 5 (e) none of the above.

4. How many of these are <u>orthonormal</u> sets of vectors?

$\{(1,2,2), (2,-2,1), (2,1,-2)\}$;　$\{(1,1), (1,-1)\}$;　$\left\{\left(\frac{1}{\sqrt{10}}, \frac{3}{\sqrt{10}}\right), \left(\frac{-3}{\sqrt{10}}, \frac{1}{\sqrt{10}}\right)\right\}$;　$\left\{\left(\frac{1}{\sqrt{6}}, \frac{-1}{\sqrt{6}}, \frac{2}{\sqrt{6}}\right), \left(0, \frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}}\right), \left(\frac{5}{\sqrt{30}}, \frac{1}{\sqrt{30}}, \frac{-2}{\sqrt{30}}\right)\right\}$

(a) 0 (b) 1 (c) 2 (d) 3 (e) none of the above.

5. If A is a singular matrix, how many of the following matrices are also singular: $A^2, A^t, AA^t, A^tA, 2A$?

(a) 1 (b) 2 (c) 3 (d) 4 (e) 5

6. If A has eigenvalues 1, 2, and 3, which of the following are the eigenvalues of $(A^2)^{-1}$?

(a) -1,-2,-3 (b) 1, 4, 9 (c) -1,-4,-9 (d) 1, 1/4, 1/9 (e) not enough information

7. How many of these are <u>linear</u> transformations? max(-3,2) = 2 (take the larger one).

$T(x, y) = \max(x,y)$;　　　$T(x, y) = (\log_2 x, -x+y)$;　　$T(x, y) = (x, y+1)$;
$T(x) = (x, 0, 2x)$;　　　$T(x, y) = (\sin(x), y)$;　　　$T(x, y) = (3x + y, 2y, x - y)$

(a) 2 (b) 3 (c) 4 (d) 5 (e) none of the above.

8. A linear transformation $T:\mathbf{R}^2 \to \mathbf{R}^2$, $T(1,1) = (1,0)$, $T(2,1) = (3,1)$. How many of the following are true?

T is one-to-one;　T is onto;　the preimage of $(2,0)$ under T is $(2,2)$;　$T^{-1}(x,y) = (x-y, x-2y)$.

(a) 1 (b) 2 (c) 3 (d) 4 (e) none of the above.

9. Find the complete solution to the linear equations: (6 points)

$$
\begin{array}{rrrrrr}
x_1 & + \; 2x_2 & - \; x_3 & - \; x_4 & = & 0 \\
x_1 & + \; 2x_2 & & + \; x_4 & = & 4 \\
-x_1 & - \; 2x_2 & + \; 2x_3 & + \; 4x_4 & = & 5
\end{array}
$$

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10. Evaluate the following determinant: (3 points)

$$\det \begin{bmatrix} 1 & -2 & 3 & 0 \\ 4 & 0 & 5 & 0 \\ 7 & -3 & 8 & 4 \\ -3 & 0 & 4 & 0 \end{bmatrix}$$

11. Compute this matrix power: $\begin{bmatrix} 1 & 5 \\ 5 & 1 \end{bmatrix}^8$ (6 points)

12. The subspace of $\mathbf{R}^3$ consisting of all vectors of the form $(b, a, a+b)$, where a and b are real numbers.
    (a) Find its basis and its dimension. (3 points)
    (b) Consider the vector $\mathbf{v} = (6,0,0)$ in $\mathbf{R}^3$. Find the closest point on the subspace to the point v. (6 points)
    (c) Find the distance between v and the subspace. (2points)

Part II. Discrete Math [50 Points]

True (T) or False (F) [20%]:
2 points for each correct answer, and −1 point for each wrong answer. Be careful.

1. $(p \wedge (\neg q)) \to (p \to q)$ is a tautology.
2.   If $\sqrt{2}$ is rational then "1>2".
3.   There exists a bijection from $N \times N$ to N.
4. $\lfloor \lceil x \rceil \rfloor = \lceil \lfloor x \rfloor \rceil$.
5.   Language $a^n b^n$ is regular.
6.   Language $a^n b^n$ cannot be accepted by a finite state machine.
7.   If $P(A|B) = 2P(A)$, then $P(B) > P(A)$.
8.   Given any random variable, say X, it is true that $Var(X) \le E(X)$.
9.   For disjoint events A and B, $P(A \cap B) = P(A) \times P(B)$.
10. For a Possion random varaible X with paramenter $\alpha=1$, the Chebysheve's inequality ensures that $P(X \ge 11) \le \frac{1}{100}$.

Muliple Choice [30%]:
Each of the following questions has aaxactly one correct choice. 2 points for each correct choice, and −0.5 point for each wrong choice. Be careful.

1.   Which of the following statements is true?
     (a)   If f_1 and f_2 are $O(g)$, then $f_1 f_2$ is $O(g)$.
     (b)   If f is $O(g)$, then f is $O(g/2)$.
     (c)   If f is $O(g)$, then g is $O(f)$.
     (d)   If f is $O(g)$, then g is not $O(f)$.
     (e)   none of the above.

2.   Which random variable of the following distributions is not a discrete one?
     (a)   exponential disribution;
     (b)   geometric distrubution;
     (c)   hypergeometric distribution;
     (d)   binomial distribution;
     (e)   none of the above.

3. $\dbinom{100}{0} + \dbinom{100}{0} \cdot 2 + \dbinom{100}{0} \cdot 2^2 + \cdots + \dbinom{100}{99} \cdot 2^{99} + \dbinom{100}{99} \cdot 2^{100} = ?$
     (a) 3^{100};
     (b) $2^{101}+1$;
     (c) 3^{101};
     (d) 3^{99};
     (e)   none of the above.

4.   The binary relation $R = \{(0, 0), (1, 1)\}$ on $A = \{0, 1, 2, 3\}$ is
     (a)   Reflexive, Not Symmetric, Transitive
     (b)   Not Reflexive, Symmetric, Transitive
     (c)   Reflexive, Symmetric, Not Transitive
     (d)   Reflexive, Not Symmetric, Not Transitive
     (e)   none of the above.

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5. Let $A = \{0, 1\} \times \{0, 1\}$ and $B = \{a, b, c\}$. Suppose A is listed in lexicographic order based on $0 < 1$ and B is in alphabetic order. If $A \times B \times A$ is listed in lexicographic order, then the next element after $((1, 0), c, (1, 1))$ is
   (a) $((1, 0), a, (0, 0))$
   (b) $((1, 1), c, (0, 0))$
   (c) $((1, 1), a, (0, 0))$
   (d) $((1, 1), a, (1, 1))$
   (e) none of the above.

6. The number of partitions of $\{1, 2, 3, 4, 5\}$ into three blocks is $S(5, 3) = 25$. The total number of functions $f : \{1,2,3,4,5\} \to \{1,2,3,4\}$ with $|\text{Image}(f)| = 3$ is
   (a) 4×6
   (b) 4×25
   (c) 25×6
   (d) $4 \times 25 \times 6$
   (e) none of the above.

7. Which of the following statements is TRUE:
   (a) For all odd integers n, $\lceil n/2 \rceil = \frac{n+1}{2}$.
   (b) For all real numbers x and y, $\lceil x+y \rceil = \lceil x \rceil + \lceil y \rceil$.
   (c) For all real numbers x, $\lceil x^2 \rceil = (\lceil x \rceil)^2$.
   (d) For all real numbers x and y, $\lceil x+y \rceil = \lfloor x \rfloor + \lfloor y \rfloor$.
   (e) none of the above.

8. The repeating decimal number $3.14159265265265 \ldots$ written as a ratio of two integers a/b is
   (a) $313845111/99990000$
   (b) $313844841/99900000$
   (c) $313845006/99990000$
   (d) $313845106/99900000$
   (e) none of the above.

9. Which of the following statements is true:
   (a) A number is rational if and only if its square is rational.
   (b) An integer n is odd if and only if $n^2 + 2n$ is odd.
   (c) A number is irrational if and only if its square is irrational.
   (d) A number n is odd if and only if $n(n + 1)$ is even
   (e) none of the above.

10. Let
   m = "Juan is a math major,"
   c = "Juan is a computer science major,"
   g = "Juan's girlfriend is a literature major,"
   h = "Juan's girlfriend has read Hamlet," and
   t = "Juan's girlfriend has read The Tempest."
   Which of the following expresses the statement "Juan is a computer science major and a math major, but his girlfriend is a literature major who hasn't read both The Tempest and Hamlet."
   (a) $c \wedge m \wedge (g \vee (\sim h \vee \sim t));$
   (b) $c \wedge m \wedge g \wedge (\sim h \wedge \sim t);$
   (c) $c \wedge m \wedge g \wedge (\sim h \vee \sim t);$
   (d) $c \wedge m \wedge g \wedge (h \vee t);$
   (e) none of the above.

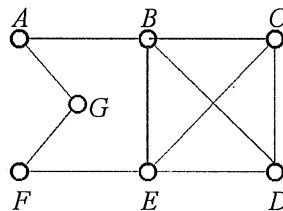11. The truth table for $(p \vee q) \vee (p \wedge r)$ is the same as the truth table for
    (a) $(p \vee q) \wedge (p \vee r)$;
    (b) $(p \vee q) \wedge r$;
    (c) $(p \vee q) \wedge (p \wedge r)$;
    (d) $p \vee q$;
    (e) none of the above.

12. How many spanning trees does the following graph have?
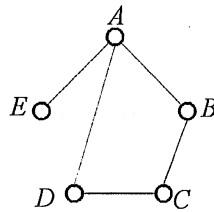
    (a) 5;
    (b) 8;
    (c) 4;
    (d) 3;
    (e) none of the above.

13. A Hamilton circuit for the following graph

    (a) must contain the edge *FE*.
    (b) must contain the edge *CD*.
    (c) must contain the edge *AB*.
    (d) all of the above;
    (e) none of the above.

14. The number of four letter words that can be formed from the letters in BUBBLE (each letter occurring at most as many times as it occurring at most as many times as it occurs in BUBBLE) is
    (a) 72;
    (b) 74;
    (c) 76;
    (d) 78;
    (e) none of the above.

15. How many different rearrangements are there of the letters in the word BUBBLE?
    (a) 60;
    (b) 80;
    (c) 100;
    (d) 120;
    (e) none of the above.