一、PART I (1~5). Multiple-choice questions (每題 4 分，共 20 分)

1. For integer n ≥ 1, r ≥ 1, and n ≥ r , which equation in the following is true？

 (A) $\binom{2n}{1}+\binom{2n}{3}+\binom{2n}{5}+\cdots+\binom{2n}{2n-1}=\binom{2n}{2}+\binom{2n}{4}+\cdots+\binom{2n}{2n}$

 (B) $\binom{2n}{n}-\binom{2n}{n-1}=\frac{1}{n+1}\binom{2n}{n}$

 (C) $\binom{n+1}{r}=\binom{n}{r-1}+\binom{n+1}{r-1}$

 (D) $\sum_{i=1}^{n}(i)(i!)=(n+1)!$

2. Let *p*, *q* and *r* be statements. The logical connectives are ∧: ***conjunction***, ∨: ***disjunction***, →: ***implication***, ¬: ***not***. Consider the following statements, which statement is **not** a tautology?

 (A) [(p ∧ q) ∨ [(p → (q → r)]] → r

 (B) p → p ∨ q

 (C) [(p → r) ∧ (q → r)] → [(p ∧ q) → r]

 (D) [p ∧ (p → q)] → q

3. For the following functions, which is **not** a one-to-one function?

 (A) f : **Z** → **Z,** f(x) = 2x + 1

 (B) f : **Q** → **Q,** f(x) = 2x + 3

 (C) f : **Z** → **Z,** f(x) = x^3 - x

 (D) f : **R** → **R,** f(x) = e^x

4. Let ∅ denote the empty set; *A*, *B* and *C* are any sets taken from a universe $\mathscr{U}$. Which statement in the following is true?

 (A) ∅ ∈ ∅

 (B) A ∩ (A ∪ B) = B

 (C) A ∩ (B ∪ C) = (A ∩ B) ∪ C)

 (D) A ∪ (A ∩ B) = A

5. Let $A=\{1, 2, 3, 4, 5, 6, 7\}$, for binary relations $\mathcal{R}$ defined on $A \times A$:
Which of the following statement is true?

(A) Define $(x, y) \in \mathcal{R}$, if x - y is a multiple of 3, then $\mathcal{R}$ is a partial order.

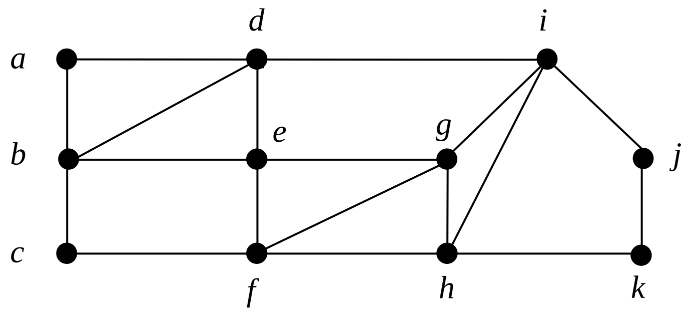
(B) Define $(x, y) \in \mathcal{R}$, if $x \le y$, then $\mathcal{R}$ is a lattice.

(C) Define $(x, y) \in \mathcal{R}$, if $x = y^2$, then $\mathcal{R}$ is a equivalence relation.

(D) Define $\mathcal{R} = \{(1,2), (2, 7), (7, 2), (1, 7)\}$, then $\mathcal{R}$ is anti-symmetric.

二、PART II (6~10). Hand-graded questions (每題 6 分，共 30 分)

6. Find an Euler circuit for the following graph if there exists.



7. An $m \times n$ zero-one matrix with m rows and n columns, such that in row i and column j, the element x_{ij} is either 0 or 1. How many 10×12 zero-one matrices have exactly one 1 in each row and at least one 1 in each column?

8. A finite state machine is a five-tuple $M = (S, \mathcal{I}, \mathcal{O}, v, w)$, where S is the set of states for M; $\mathcal{I}$ is the input alphabet for M, $\mathcal{O}$ is the output alphabet for M, v: $S \times \mathcal{I} \to S$ is the next state function, w: $S \times \mathcal{I} \to \mathcal{O}$ is the output function. Construct a finite state machine M with $\mathcal{I} = \mathcal{O} = \{0, 1\}$ that recognizes the occurrence of 101 for any input string $\{0,1\}^*$. For example, the input string: `101011110101`, then the output string: `001010000101`.

9. Define the integer sequence $a_0, a_1, a_2, a_3, \ldots$, recursively by

(1) $a_0 = 1, a_1 = 1, a_2 = 1$; and

(2) For $n \ge 3$, $a_n = a_{n-1} + a_{n-3}$.

Prove that $a_{n+2} \ge (\sqrt{2})^n$ for all $n \ge 0$.

10. Let $S \subseteq \{1, 2, 3, 4, 5, \ldots, 9\}$ and $|S| = 5$. Prove that the sums of the elements in all the nonempty subsets of S cannot all be distinct.

三、PART III (11~15). Hand-graded questions (每題 10 分，共 50 分)

11.If A and B are 3×3 matrices and $|A| = -1$ and $|B| = 2$, compute the following determinants. (a) $\left|3A^2B^{-1}\right|$ (b) $\left|\left(2AB^t\right)^{-1}\right|$, where B^t represents the transpose of matrix B.

12.Solve the following system of equations using the method of Gauss-Jordan elimination.

$$\begin{cases} x_1 + x_2 + x_3 - x_4 = -3 \\ 2x_1 + 3x_2 + x_3 - 5x_4 = -9 \\ x_1 + 3x_2 - x_3 - 6x_4 = -7 \\ -x_1 - x_2 - x_3 = 1 \end{cases}$$

13. Consider the linear transformation $T(x, y) = (3x + 4y, x + 2y)$ of $R^2 \to R^2$.

(a) Prove that T is invertible and find the inverse of T.

(b) Determine the pre-image of the vector (1, 1).

14.Let $u = (x_1, x_2)$ and $v = (y_1, y_2)$ be elements of R^2. Prove that the following function defines an inner product on R^2: $\langle u, v\rangle = 4x_1y_1 + 9x_2y_2$.

15.Find a matrix P that orthogonally diagonalizes the following matrix A, and determine $P^{-1}AP$.

$$A = \begin{bmatrix} 3 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 3 \end{bmatrix}.$$