國立高雄大學 102 學年度研究所碩士班招生考試試題

科目：線性代數
考試時間：100 分鐘

系所：應用數學系
身份別：一般生、在職生
本科原始成績：100 分

是否使用計算機：否

Notations.

P_n : the set of polynomials of degree at most n.
$M_{n\times m}(\mathbb{R})$: the set of $n \times m$ real matrices.
$\mathrm{Ker}(T)$: the null space of a linear transformation T.

1 Let $A = \begin{bmatrix} 1 & 3 & 0 & -1 & 2 \\ 0 & -2 & 4 & -2 & 0 \\ 3 & 11 & -4 & -1 & 6 \\ 2 & 5 & 3 & -4 & 0 \end{bmatrix}$.

a. (5) Find a basis for the column space.
b. (5) Find a basis for the row space.
c. (5) Find a basis for the null space.

2 Let $A, B \in M_{n\times n}(\mathbb{R})$

a. (8) Prove that $\mathrm{rank}(AB) \leq \mathrm{rank}(A)$.
b. (7) Prove that $\mathrm{rank}(A^T A) = \mathrm{rank}(A)$.

3 Let

$$A = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}.$$

a. (10) Find the eigenvalues and eigenvectors of A.
b. (5) Compute $\lim_{m\to\infty} B_m$, where $B_m = I + A + \frac{A^2}{2!} + \cdots + \frac{A^m}{m!}$.

4 Let $A = \begin{bmatrix} 3 & 1 \\ 1 & 2 \\ 2 & -1 \end{bmatrix}$, $b = \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}$ and $c = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

a. (10) Find the solution of $\min_{x\in\mathbb{R}^2} \|Ax - b\|$.
b. (10) Find the solution of $\min_{A^T y = c} \|y\|$.

5 Let $T : P_2 \to P_3$ be defined by $T(p(x)) = (x+1)p(x-1)$.

a. (5) Show that T is a linear transformation.
b. (10) Let $\beta = \{x^2, x, 1\}$ and $\beta' = \{x^3, x^2, x, 1\}$ be the ordered bases for P_2 and P_3, respectively. Find the matrix representation $[T]_\beta^{\beta'}$.

6 (10) Show that a linear transformation $T : V \to W$ is one-to-one if and only if $\mathrm{Ker}(T) = \{0\}$.

7 (10) Let $A = A^T \in M_{n\times n}(\mathbb{R})$. Show that eigenvectors of A that correspond to different eigenvalues are orthogonal.