國立中央大學101學年度碩士班考試入學試題卷

所別：數學系碩士班 甲組(一般生)　科目：線性代數　共 1 頁 第 1 頁
數學系碩士班 甲組(在職生)
數學系碩士班 乙組(一般生)

本科考試禁用計算器　　＊請在試卷答案卷（卡）內作答

1. (10 points) Let A be a 3×3 real matrix with characteristic polynomial $x^3 - x$. Then find a diagonal matrix similar to A. Prove your answer.

2. (10 points) Let

$$A = \begin{bmatrix} 1 & 8 & -2 \\ 0 & 2 & 4 \\ 0 & 0 & 3 \end{bmatrix}.$$

Express A^{-1} as a polynomial of A. Prove your answer.

3. (10 points) Let A be the real matrix

$$A = \begin{bmatrix} 2 & 0 & 0 \\ a & 2 & 0 \\ b & c & -1 \end{bmatrix}.$$

Find the values of the parameters a, b and c for which the matrix A is diagonalizable. Prove your answer.

4. (10 points) Let

$$A = \begin{bmatrix} 0.4 & 0.2 & 0.3 \\ 0.2 & 0.4 & 0.3 \\ 0.4 & 0.4 & 0.4 \end{bmatrix}.$$

Find $\lim_{n\to\infty} A^n$. Prove your answer.

5. (10 points) Let A be the matrix

$$\begin{bmatrix} s & -1 & 0 \\ -1 & s & -1 \\ 0 & -1 & s \end{bmatrix},$$

where $s \in \mathbb{R}$ is a parameter. Find the values of s for which the pairing on the column vectors in $\mathbb{R}^3$

$$\langle v, w \rangle = v^T A w$$

is an inner product. Prove your answer.

6. (10 points) Suppose that $A = -A^T$ (such a matrix A is called *skew-symmetric*). Show that $I + A$ is invertible, where I is the identity matrix of the same dimension as A.

7. (20 points) Let A be an $n \times n$ matrix of rank r where $0 < r < n$ and let A be partitioned as

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix},$$

where A_{11} is an $r \times r$ matrix of rank r, and A_{12} is an $r \times (n-r)$ matrix, and A_{21} is an $(n-r) \times r$ matrix, and A_{22} is an $(n-r) \times (n-r)$ matrix. Prove that $A_{22} = A_{21} A_{11}^{-1} A_{12}$.

8. (20 points) If $f_1, f_2, \ldots, f_n$ are continuous functions from the interval $[0,1]$ to $\mathbb{R}$, all different from the zero-function, and satisfying $\int_0^1 f_i(x) f_j(x) dx = 0$ for all $i, j \in \{1, 2, \ldots, n\}$ with $i \neq j$. Prove that $f_1, f_2, \ldots, f_n$ are linearly independent.