Linear Algebra

1. The 3×3 symmetry matrix $\boldsymbol{A}$ has 3 eigenvalues {0, 1, 2} and corresponding eigenvectors

$$\begin{bmatrix}1\\0\\-1\end{bmatrix},\begin{bmatrix}0\\1\\0\end{bmatrix},\begin{bmatrix}1\\0\\1\end{bmatrix}.$$

(a) (5 %) Find A where each entry is integer.

(b) (10 %) Find the LU factorization of $\boldsymbol{A}$.

2. Let $\boldsymbol{B} = \{[1\ 0\ 0]^T, [0\ 1\ 0]^T, [0\ 0\ 1]^T\}$ and $\boldsymbol{B}' = \{[1\ 1\ 0]^T, [1\ 0\ 1]^T, [0\ 1\ 1]^T\}$ be bases for $\boldsymbol{R}^3$, and let

$$M = \begin{bmatrix}1 & 0 & 0\\0 & 2 & 0\\0 & 0 & 3\end{bmatrix}$$

be the matrix for $\boldsymbol{T}: \boldsymbol{R}^3 \rightarrow \boldsymbol{R}^3$ relative to $\boldsymbol{B}$.

(a) (5 %) Find the transition matrix $\boldsymbol{P}$ from $\boldsymbol{B}'$ to $\boldsymbol{B}$.

(b) (5 %) Use the matrices $\boldsymbol{M}$ and $\boldsymbol{P}$ to find $[T(\boldsymbol{v})]_{\boldsymbol{B}}$, where $[\boldsymbol{v}]_{\boldsymbol{B}'} = [1\ 0\ -1]^T$.

(c) (5 %) Find $\boldsymbol{P}^{-1}$.

(d) (5 %) Find the matrix $\boldsymbol{M}'$ for $\boldsymbol{T}$ relative to $\boldsymbol{B}'$.

(e) (5 %) Find $[T(\boldsymbol{v})]_{\boldsymbol{B}'}$.

3. An augmented matrix for a linear system can be expressed as

$$\left[\begin{array}{ccc|c}0 & 1 & 1 & 1\\2 & 3 & 7 & 5\\1 & 3 & a & b\end{array}\right]$$

(a) (5 %) Find a and b for the inconsistent system.

(b) (5 %) Find a and b for the system with infinitely many solutions.

Probability

1. (10 %)

Let X and Y be two random variables. The probability density function (pdf) of X is denoted as $f_X(x)$ while the pdf of Y is denoted as $f_Y(y)$. Let a new random variable Z be defined as $Z = XY$.
Proof that the pdf of Z can be expressed as

$$f_Z(z) = \int_{-\infty}^{\infty} \frac{f_Y(z/x) f_X(x)}{x} dx = \int_{-\infty}^{\infty} \frac{f_X(z/y) f_Y(y)}{y} dy$$

2. (10%)

Let X be a continuous random variable with characteristic function $\Psi_X(\omega)$. Let a new random variable Y be defined as $Y = aX + b$, where a and b are two scalar constants. Find the characteristic function of Y.

3.

The joint cumulative distribution function (cdf) of two random variables X and Y is given by

$$F_{X,Y}(x,y) = \begin{cases} 0 & , x < 0 \text{ or } y < 0 \\ p_1 & , 0 \le x < a,\ 0 \le y < b \\ p_2 & , x \ge a,\ 0 \le y < b \\ p_3 & , 0 \le x < a,\ y \ge b \\ 1 & , x \ge a,\ y \ge b \end{cases}$$

(a) Find the marginal cdf of X. (5%)

(b) Find the marginal cdf of Y. (5%)

(c) Find the conditions on p_1, p_2, and p_3 for which X and Y are independent. (10%)

4.

A company producing USB drives has three manufacturing plants producing 50, 30, and 20 percent respectively, of its product. Suppose that the probabilities that a USB drive manufactured by these plants is defective are 0.02, 0.05, and 0.01, respectively.

(a) If a USB drive is selected at random from the output of the company, what is the probability that it is defective? (5%)

(b) If a USB drive is selected at random is found to be defective, what is the probability that it was manufactured by plant 2? (5%)