1. (10%) Give Taylor's series expansion of the function $f(x)$ about x_0.

2. (10%) Solve the following system of linear equations by the Gauss elimination:

$$\begin{cases} x+ \ y+ \ z =-1 \\ \quad\ 4y+ 6z = \ 6 \\ \quad\ \ y+ \ z = \ 1 \end{cases}$$

3. (15%) Find the area of the parallelogram of which the given vectors [1, 0, 2] and [1, 1, 2] are adjacent sides.

4. (15%) Find the eigenvalues and eigenvectors of the matrix $\mathbf{A}$:

$$\mathbf{A} = \begin{bmatrix} 5 & 10 \\ 4 & -1 \end{bmatrix}.$$

5. (10%) Solve the Ordinary Differential Equation

$$xy'' - y' + 4x^3 y = 0.$$

6. (10%) Suppose $f(t)$ satisfies the difference — differential equation

$$\frac{df(t)}{dt} + f(t) - f(t-1) = 0 \qquad (t \geq 0)$$

and the initial condition, $f(t) = f_0(t)$, $-1 \leq t \leq 0$, where $f_0(t)$ is given.
Show that the Lapalace transform $F(s)$ of $f(t)$ satisfies

$$F(s) = \frac{f_0(0)}{1+s-e^{-s}} + \frac{e^{-s}}{1+s-e^{-s}} \int_{-1}^{0} e^{-st} f_0(t)\,dt.$$

Find $f(t)$, $t \geq 0$, when $f_0(t) = 1$. Check the result.

7. (15%) Find the Fourier coefficients of the periodic function $f(x)$ in Fig. 1. The formula is

$$f(x) = \begin{cases} -k & \text{if } -\pi < x < 0 \\ k & \text{if } 0 < x < \pi \end{cases} \text{ and } f(x + 2\pi) = f(x).$$

Functions of this kind occur as external forces acting on mechanical systems, wave systems, electromotive forces in electric circuits, etc. (The value of $f(x)$ at a single point does not affect the integral; hence we can leave $f(x)$ undefined at $x = 0$ and $x = \pm\pi$.)
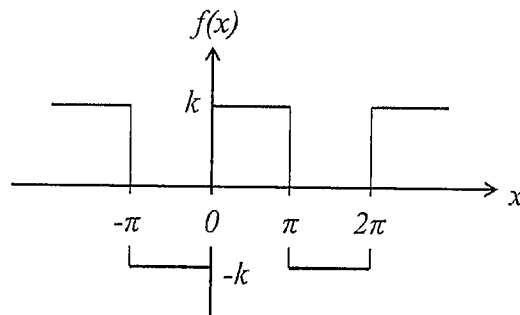

Fig. 1

8. (15%) A sinusoidal voltage $E \sin \omega t$, where t is time, is passed through a half-wave rectifier that clips the negative portion of wave (Fig. 2). Find the Fourier series of the resulting periodic function

$$u(t) = \begin{cases} 0 & \text{if } -L < t < 0 \\ E \sin \omega t & \text{if } 0 < t < L \end{cases}, \quad L = \frac{\pi}{\omega}$$
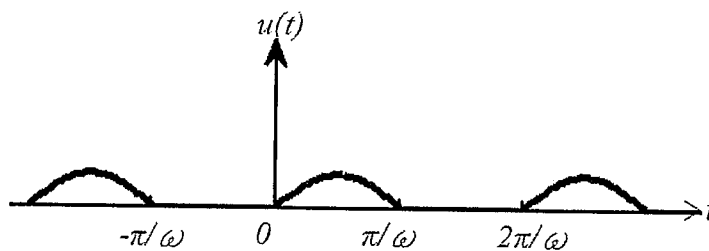

Fig. 2