國立中山大學102學年度碩士暨碩士專班招生考試試題

科目名稱：機率【通訊所碩士班甲組】 題號：437004

※本科目依簡章規定「可以」使用計算機（廠牌、功能不拘） 共2頁第1頁

1. (Totally, 10 pts) The conditional probability density of X provided that the continuous event V has values between y and $y+dy$ is given by

$$p_{X|Y}(x\,|\,y)=\frac{2xy+a}{a^2(y+1)}\text{ for }0\le x\le a,$$

Where a is a constant and the probability density of Y is given by

$$p_Y(y)=\frac{2(y+1)}{b^2+2b}\text{ for }0\le y\le b.$$

Please find the conditional probability density function $p_{Y|X}(y\,|\,x)$, i.e., the conditional probability density function of Y provided that the continuous event U has values between x and $x+dx$.

2. (Totally, 10 pts) Let Z_1 and Z_2 be independent and have exponential distribution with density $\lambda e^{-\lambda z}$ for $z\ge 0$. Define $X=Z_2$ and $Y=Z_1+Z_1Z_2$. Please find $E[E[Y|X]]$.

3. (Totally, 15 pts) Markov Inequality is expressed as follows. Let Y be a non-negative random variable with finite expectation $E[Y]=\eta$, then, for any $\alpha>0$,

$$P\{Y>\alpha\}\le\frac{\eta}{\alpha}.$$

(a) (5 pts) Please prove Chernoff Bound using Markov Inequality. Note that Chernoff Bound is given by, for a random variable X,

$$P\{X>\alpha\}\le\frac{E[e^{sX}]}{e^{s\alpha}},\text{ for }s>0$$

(b) (10 pts) The characteristic function of a Gaussian random variable X distributed as $N(\mu,\sigma^2)$ is given by

$$\Phi_X(\omega)=e^{j\mu\omega-\frac{\sigma^2\omega^2}{2}}.$$

Please find the Chernoff Bound for the above Gaussian random variable.

4. (Totally, 15 pts) Let X and Y be independent random variables each Poisson distributed with parameter λ.
(a) (5 pts) Find the probability mass function of $X+Y$.
(b) (5 pts) Find the distribution function of $\min(X,Y)$.
(c) (5 pts) Find the conditional probability $P(Y=y|X+Y=z)$ for $y=0,1\ldots,z$.

5. (Totally, 15 pts) Consider a random variable X with the following PDF

$$p(x)=\tfrac{3}{2}x^2,\qquad\text{for }-1\le x\le 1$$

(a) (5pts) Plot the cumulative distribution function (CDF) of X
(b) (5pts) Find the expectation $E[X]$
(c) (5pts) Find the variance of X

6. (Totally, 25 pts) Let X and Y be two random variables with the following joint PMF:

$$P(X,Y)=\begin{cases}\frac{3}{10}, & X=1,Y=2\\ \frac{1}{10}, & X=1,Y=4\\ \frac{1}{5}, & X=2,Y=2\\ \frac{2}{5}, & X=2,Y=4\end{cases}$$

(a) (5pts) Find the conditional probability $P(X|Y=2)$.
(b) (5pts) Find the conditional expectation $E[X|Y=4]$
(c) (5pts) Are X and Y independent of each other? Please prove your answer.
(d) (10pts) Find the correlation coefficient between X and Y.

7. (Totally, 10 pts) Let U be a random variable uniformly distributed between 0 and 1. Answer the following questions.
(a) (5pts) For any strictly increasing function $f:\mathbb{R}\rightarrow[0,1]$, find the CDF of $X=f^{-1}(U)$.
(b) (5 pts) Given any random variable X with PDF $p(x)$, show that X can be generated by $X=f^{-1}(U)$, where

$$f(x)=\int_{-\infty}^{x}p(v)\ dv.$$