系所組別：水利及海洋工程學系甲、乙組

考試科目：工程數學

考試日期：0223，節次：3

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1. Consider functions f and g that satisfy Laplace equation ($\nabla^2 f = \nabla^2 g = 0$) in some domain D containing a region T with boundary surface S such that T satisfies the assumptions in the divergence theorem. Prove

   (a) (8%) $\iint\limits_S g\frac{\partial g}{\partial n} dA = \iiint\limits_T |\text{grad}g|^2 \, dV$

   (b) (4%) If $\frac{\partial g}{\partial n} = 0$ on S, then g is constant in T.

   (c) (8%) $\iint\limits_S \left(f\frac{\partial g}{\partial n} - g\frac{\partial f}{\partial n}\right) dA = 0.$

   Note: grad$g$ means ∇g.

2. (16%) Evaluate the integral $I = 2\int_0^\infty \frac{\sin x}{x(a^2+x^2)} dx$ with $a > 0$ using contour integral.

3. (a) (12%) Find the solution of the initial-value problem

   $\frac{\partial^2 u}{\partial t^2} = c^2 u_{xx}$ $-\infty < x < \infty$, c is a constant.

   $u(x,0) = f(x)$

   $u_t(x,0) = g(x)$

   This problem which has no boundaries describes the motion of an infinite string with given initial conditions and was solved by D'Alembert.

   (b) (8%) obtain the solution $u(x,t)$ if

   $f(x) = \begin{Bmatrix} 2x & \text{if } 0 < x \le \frac{1}{2} \\ 2(1-x) & \text{if } \frac{1}{2} < x < 1 \end{Bmatrix}$, $g(x) = 0$, plot the displacement diagram in space at time $t = 0$, $\frac{1}{2c}$, $\frac{1}{c}$.

4. In steady equilibrium, the temperature field $T(r,\theta)$ in the circle $r < a$ is governed by Laplace's equation which can be expressed as

   $\nabla^2 T = T_{xx} + T_{yy} = T_{rr} + \frac{1}{r}T_r + \frac{1}{r^2}T_{\theta\theta} = 0$ for $r < a$

   If the temperature on the circumference can be specified as

   $T(a,\theta) = f(\theta)$, $0 \le \theta \le 2\pi$

   (a) (12%) Solve this problem using separation of variables.

   (b) (8%) The series obtained from (a) can be summed explicitly, which is known as Poisson's formula. Find this formula.

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5. (24%) Determine whether each statement is true or false related to linear algebra. If a statement is true, give a reason or cite an appropriate statement. If a statement is false, provide an example that shows that the statement is not true in all cases or cite an appropriate statement. (each question weighs 3%)

(a) If the determinant of an $n \times n$ matrix $\mathbf{A}$ is nonzero, the $\mathbf{Ax} = 0$ has only the trivial solution.

(b) An invertible square matrix $\mathbf{A}$ is called orthogonal if $\mathbf{A}^{-1} = \mathbf{A}^T$ Then $\det(\mathbf{A}) = \pm\mathbf{1}$.

(c) If $\mathbf{x}$ is the eigenvector of $\mathbf{Ax} = \lambda\mathbf{x}$ with λ being eigenvalue, then the determinant of $\mathbf{A} - \lambda\mathbf{I}$ is zero.

(d) If $\mathbf{A}$ and $\mathbf{B}$ are nonsingular $n \times n$ matrices, then $\mathbf{A} + \mathbf{B}$ is a nonsingular matrix.

(e) For any matrix $\mathbf{A}$, the matrix $\mathbf{AA}^T$ is symmetric.

(f) If the matrices $\mathbf{A}, \mathbf{B}$, and $\mathbf{C}$ satisfy $\mathbf{AB} = \mathbf{AC}$, then $\mathbf{B} = \mathbf{C}$.

(g) If $\mathbf{A}$ can be row reduced to the identity matrix, then $\mathbf{A}$ is nonsingular.

(h) If $\mathbf{A}$ is an $n \times n$ matrix, then $\mathbf{A}$ is orthogonally diagonalizable and has real eigenvalues.

Note: $\det(\mathbf{A})$ stands for determinant of matrix $\mathbf{A}$, $\mathbf{A}^T$ is the transpose of $\mathbf{A}$, $\mathbf{A}^{-1}$ is the inverse of $\mathbf{A}$, and $\mathbf{I}$ is identity matrix.