※ 考生請注意：本試題不可使用計算機

1.

If the equation of motion of a system can be described as:

$\frac{d^2x}{dt^2}+\omega^2 x = F_0 \cos\gamma t$, where F_0 is constant and $r \neq \omega$, find

(a) Wronskian = ? (5%)

(b) general solution $x(t)$ under initial conditions: $x(0)=0$ and $\dot{x}(0)=0$ (15%)

(c) $\lim\limits_{\gamma\to\omega} x(t)$ (5%)

2. (1) Find the eigenvalues and corresponding eigenvectors of the

matrix $\mathbf{A} = \begin{pmatrix} -2 & 4 \\ -1 & 3 \end{pmatrix}$ and also show that $\mathbf{A}^{100} = \begin{pmatrix} \frac{-2^{100}+4}{3} & \frac{2^{102}-4}{3} \\ \frac{-2^{100}+1}{3} & \frac{2^{102}-1}{3} \end{pmatrix}$. (15 %)

(2) Find the equation of the tangent plane to the graph of

$x^2 - 4y^2 + z^2 = 36$ at point (2, 1, 6). (10 %)

(背面仍有題目,請繼續作答)

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3. (a) Solve the wave equation

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}$$

with

$$u(x,0) = \sin(\pi x), \quad \frac{\partial u}{\partial t}(x,0) = 0, \quad u(0,t) = 0 \quad \text{and} \quad u(1,t) = 0. \quad \textbf{(15\%)}$$

(b) Solve the heat conduction equation

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t} + \sin(\pi x)$$

with

$$u(x,0) = \sin(2\pi x), \quad u(0,t) = 0 \quad \text{and} \quad u(1,t) = 0. \quad \textbf{(10\%)}$$

4. (1) Calculate the **Cauchy principal value** of the integral

$$\int_{-\infty}^{\infty} \frac{\sin x}{x(x^2+1)}\,dx. \quad \textbf{(15\%)}$$

(2) The mapping

$$w = z + \frac{k^2}{z}$$

with k being a positive constant is called the **Joukowski transformation**. Show that the Joukowski transformation maps any circle $x^2 + y^2 = R^2$ into an ellipse when $k \neq R$. **(6%)** Also determine the image of the circle when $k = R$. **(4%)**