※ 考生請注意：本試題不可使用計算機

[1] (A total of 20 points)

(a) (3 points) An equation of motion of one dimensional oscillator is given by

$$m\frac{d^2x}{dt^2} = -kx, \tag{1}$$

where t is time, $x(t)$ is the position of the point mass m, and k is the spring constant. Normalizing time by $\sqrt{m/k}$ and length by the oscillation amplitude A, obtain a dimensionless equation

$$\frac{d^2X}{dT^2} = \ddot{X} = -X. \tag{2}$$

Here, the *dot* operator stands for d/dT, $T = (\sqrt{k/m})t$, and $X = x/A$. Using the initial conditions $\dot{X}(T = 0) = 1$ and $X(T = 0) = 0$, solve Eq.(2) for $X(T)$.

(b) (7 points) An equation of motion of a damping oscillator is given by

$$m\frac{d^2x}{dt^2} = -kx - \nu\frac{dx}{dt}, \tag{3}$$

where ν is a friction constant. As in **(a)**, normalize Eq.(3) to obtain

$$\ddot{X} = -X - \epsilon X. \tag{4}$$

What is ϵ in terms of m, k, and ν? Using the initial conditions $\dot{X} = 1$ and $X = 0$, solve Eq.(4) to obtain an exact solution $X_{exact}(T)$.

(c) (5 points) Let us solve **(b)** by a perturbation method. By assuming $\epsilon \ll 1$, we apply an expansion

$$X = X_0 + \epsilon X_1 + \epsilon^2 X_2 + ... \tag{5}$$

to Eq.(4). We separate the equations order by order in terms of a small parameter ϵ in the perturbation method. For example, at the lowest order $[O(1)]$, we obtain

$$\ddot{X}_0 = -X_0. \tag{6}$$

At the order ϵ $[O(\epsilon)]$, we obtain

$$\ddot{X}_1 = -X_1 - X_0, \tag{7}$$

where the solution of X_0 from Eq.(6) is substituted to the right side of Eq.(7). Solve Eq.(6) for $X_0(T)$ using the initial conditions $\dot{X}_0 = 1$ and $X_0 = 0$. Solve Eq.(7) for $X_1(T)$ using the initial conditions $\dot{X}_1 = 0$ and $X_1 = 0$.

(d) (5 points) Continue the expansion of Eq.(4) to obtain the equation at $O(\epsilon^2)$. Solve the $O(\epsilon^2)$ equation for $X_2(T)$ using initial conditions $\dot{X}_2 = 0$ and $X_2 = 0$. By adding $X_{pert} = X_0 + \epsilon X_1 + \epsilon^2 X_2$, obtain the approximate solution X_{pert}. Compare the solution with the exact solution in **(b)** (hint: do a Taylor expansion of X_{exact}). To what order in ϵ is the approximate solution correct?

(背面仍有題目,請繼續作答)

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[**2**] (A total of 25 points)

(a) (10 points) A one dimensional heat conduction equation is given by

$$\frac{\partial T(x,t)}{\partial t} = \frac{\partial^2 T(x,t)}{\partial x^2} \tag{8}$$

in a domain $0 \leq x \leq 1$ and $t \geq 0$.

As suggested in Fig.2, the initial condition is given by

$$T(x,0) = 1 - x^2. \tag{9}$$

Take boundary conditions $T(0,t) = 1$ and $T(1,t) = 0$. Draw the expected profile of T at $t \to \infty$. Solve Eq.(8) for $T(x,t)$.

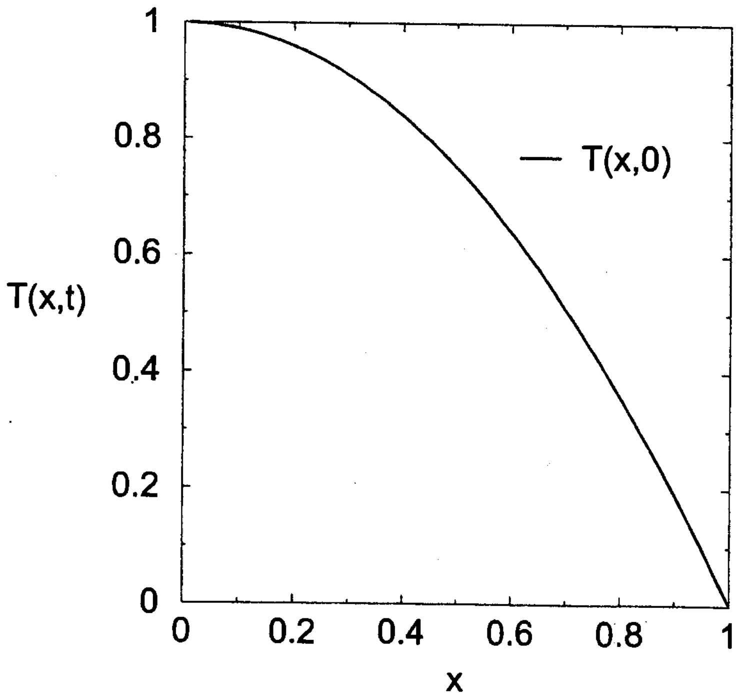
(b) (10 points) Solve Eq.(8) for $T(x,t)$ by taking the initial condition Eq.(9), and boundary conditions $\partial_x T(0,t) = 0$ and $T(1,t) = 0$. Draw the expected profile of T at $t \to \infty$.

(c) (5 points) In **(b)**, we would like to find a steady state solution (means the T profile do not evolve with time) by adding a constant source term S

$$\frac{\partial T(x,t)}{\partial t} = \frac{\partial^2 T(x,t)}{\partial x^2} + S. \tag{10}$$

Obtain the value of S.

Fig.2



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[3] (A total of 15 points)

(a) (5 points) Consider a one-way wave equation,

$$\frac{\partial}{\partial t}u(x,t) = -c\frac{\partial}{\partial x}u(x,t). \tag{11}$$

where $c > 0$ is a constant. Show that $u(x,t) = f(x-ct)$ can be a solution in general.[1] Does the wave move to the right or to the left, and why? The lines given by $x - ct = constant$ are called characteristic curves. Draw the characteristic curves taking x as the abscissa and t as the ordinate.
Take $c = 1$ for the moment. If a wave form of

$$u(x,0) = \begin{cases} \sin(\pi x) & \text{for } 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases} \tag{12}$$

is taken as an initial condition, what will be the wave shape at $t = 1$? Draw the wave shape in a 3D figure within $0 \le x \le 4$ (see Fig.3) and comment on how the wave propagates.

(b) (5 points) We replace c by $2t$ in Eq.(11). Draw the characteristic curves on the xt-plane. As in (a), taking Eq.(12) as an initial condition, draw the wave shape at $t = 1$ in a 3D figure within $0 \le x \le 4$.

(c) (5 points) Finally, we replace c by $u(x,t)$ in Eq.(11).[2]

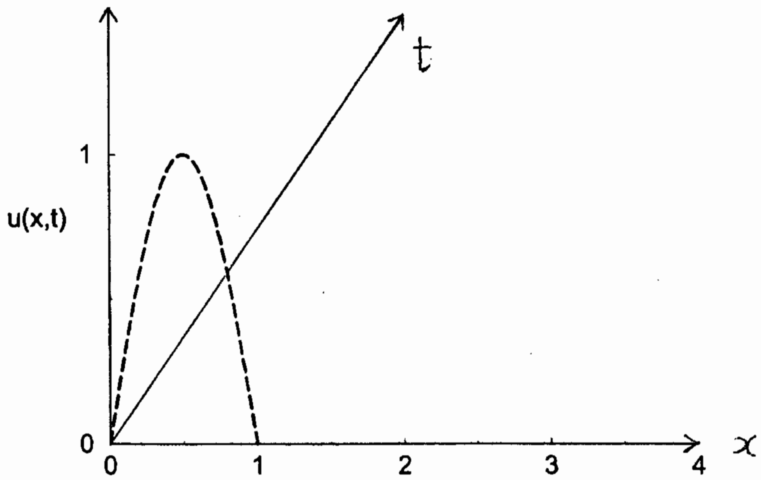
$$\frac{\partial}{\partial t}u(x,t) = -u(x,t)\frac{\partial}{\partial x}u(x,t).$$

Show that in this nonlinear case, $u(x,t) = g(x-ut)$ can be a solution in general. We take

$$u(x,0) = \begin{cases} 2 & \text{for } x < 0 \\ 2-x & \text{for } 0 \le x \le 1 \\ 1 & \text{for } x > 0 \end{cases}$$

as an initial condition. Draw the characteristic curves on the xt-plane. Draw the shape of $u(x,t)$ at $t = 0$ and $t = 1$ in a 3D figure within $0 \le x \le 4$. Comment on what happens after $t = 1$.

Fig.3



[1]The familiar sinusoidal form, $f(x-ct) = \sin(x-ct)$ is one of the solutions.
[2]The right side corresponds to a *convective derivative*.

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[4] (A total of 20 points)

(a) (4 points) Integrate $f(z) = (z-\alpha)^n$ along a circle C (radius r, centered at α). Prove

$$\int_C (z-\alpha)^n dz = \begin{cases} 2\pi i & \text{for } n = -1 \\ 0 & \text{for } n \neq -1 \end{cases}.$$

Here, z and α are complex numbers and $i = \sqrt{-1}$ is the imaginary unit.

(b) (4 points) Prove Cauchy's integral theorem

$$\int_C f(z)dz = 0.$$

Here, $f(z)$ is a holomorphic function in a complex domain D, while C is a closed loop within D. You can use the Cauchy-Riemann differential equations for the proof.

(c) (4 points) Prove Cauchy's integral formula

$$f(\alpha) = \frac{1}{2\pi i}\int_C \frac{f(z)}{z-\alpha}dz.$$

Here, $f(z)$ is a holomorphic function in a complex domain D, and C is a closed "counter-clockwise" loop within D. Employing Cauchy's integral formula estimate

$$I_1 = \int_\gamma \frac{\cos(z)}{z+i}dz$$

where γ is a circle of radius 1 with its center located at $z = -i$.

(d) (4 points) When $f(z)$ is expanded in the form

$$f(z) = \sum_{k=-\infty}^{\infty} a_k (z-\alpha)^k$$

the coefficient "a_{-1}" is called the *residue* of $f(z)$ at α, which is described by "$\text{Res} f(z)|_{z=\alpha}$". Prove Cauchy's residue theorem

$$\int_C f(z)\,dz = 2\pi i \sum_{k=1}^{m} \text{Res} f(z)|_{z=\alpha_k}$$

where m is the number of isolated singular points α_k inside the loop C.

(e) (4 points) Using the residue theorem estimate

$$I = \int_{-\infty}^{\infty} \frac{1}{x^2+1}dx.$$

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[5] (A total of 20 points)

(a) (6 points) Consider a system of simultaneous ordinary differential equations given by

$$\frac{dx}{dt} = y + z,$$
$$\frac{dy}{dt} = z + x,$$
$$\frac{dz}{dt} = x + y.$$

We would like to write the three equations in a matrix form

$$\frac{d\mathbf{u}}{dt} = \mathbf{A} \cdot \mathbf{u},$$

where $\mathbf{u}^t = (x, y, z)$ and $\mathbf{A}$ is a 3×3 matrix. Write the nine components of $\mathbf{A}$ explicitly. Obtain eigenvalues and eigenvectors of the matrix $\mathbf{A}$.

(b) (6 points) Is the matrix $\mathbf{A}$ diagonalizable? If yes, find a matrix $\mathbf{P}$ which satisfies $\mathbf{P}^t\mathbf{A}\mathbf{P} = \mathbf{D}$. Here, $\mathbf{P}^t$ is a transpose of $\mathbf{P}$ and $\mathbf{D}$ is a diagonal matrix.

(c) (4 points) Obtain the solutions for $x(t)$, $y(t)$, and $z(t)$ taking initial conditions $(x, y, z)|_{t=0} = (1, 1, 1)$.

(d) (4 points) Obtain the solutions for $x(t)$, $y(t)$, and $z(t)$ taking initial conditions $(x, y, z)|_{t=0} = (-2, 1, 1)$. What are the solutions at $t \to \infty$?