考生注意: 本試題不可使用計算機

Notation.

- For any field F, (1) F^n is the n-dimensional vector space over F, (2) $F^{m\times n}$ is the set of all $m \times n$ matrices over F, and (3) if $A \in F^{m\times n}$, then A^T denotes the transpose of A.
- $\mathbb{R}$: the field of real numbers.
- $\mathbb{C}$: the field of complex numbers.
- For a matrix $A \in \mathbb{C}^{m\times n}$, A^H denotes the conjugate transpose of A.
- For a vector $z \in \mathbb{C}^n$, z^H denotes the conjugate transpose of z.

[15%] 1. For an element $\boldsymbol{z} \in \mathbb{R}^3$, we denote by z_1, z_2, and z_3 the coordinates of $\boldsymbol{z}$. That is, $\boldsymbol{z} = (z_1, z_2, z_3)$. Which of the following subsets of $\mathbb{R}^3$ are actually subspaces? Give your reasons. [Note. There will be no points given if no reasons are given.]

(a) The set of vectors $\boldsymbol{z}$ with $z_1 = z_2$.

(b) The set of vectors $\boldsymbol{z}$ with $z_1 = 1$.

(c) The set of vectors $\boldsymbol{z}$ with $z_1 z_2 z_3 = 0$.

(d) All vectors $\boldsymbol{z}$ that satisfy $z_1 + z_2 + z_3 = 0$.

(e) All vectors $\boldsymbol{z}$ with $z_1 \le z_2 \le z_3$.

[10%] 2. Let V be the vector space over $\mathbb{R}$ spanned by the vectors $\boldsymbol{a} = (1, -1, 0, 0)$, $\boldsymbol{b} = (0, 1, -1, 0)$, and $\boldsymbol{c} = (0, 0, 1, -1)$. Find an orthonormal basis for V.

[15%] 3. For $n \ge 2$, let F_n be the determinant of the $n \times n$ tri-diagonal matrix

$$\begin{bmatrix} 1 & -1 & & & \\ 1 & 1 & -1 & & \\ & 1 & 1 & \ddots & \\ & & \ddots & \ddots & -1 \\ & & & 1 & 1 \end{bmatrix} \in \mathbb{R}^n.$$

For example, $F_2 = \det \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} = 2$ and $F_3 = \det \begin{bmatrix} 1 & -1 & 0 \\ 1 & 1 & -1 \\ 0 & 1 & 1 \end{bmatrix} = 3$. Also set $F_0 = 1$ and $F_1 = 1$.

(a) Show that $F_n = F_{n-2} + F_{n-1}$ for $n \ge 2$.

(b) Let $u_n = \begin{bmatrix} F_{n-1} \\ F_n \end{bmatrix}$, $n = 1, 2, \ldots$. Then $u_n = Au_{n-1}$. What is A?

(c) Evaluate F_{100}.

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[15%] 4. Let A be the Markov matrix $A = \begin{bmatrix} 0.8 & 0.05 \\ 0.2 & 0.95 \end{bmatrix} \in \mathbb{R}^2$. Let $u_0 = \begin{bmatrix} 0.02 \\ 0.98 \end{bmatrix}$. For integers $k \geq 1$, define $u_k = A^k u_{k-1}$. Find $\lim_{k\to\infty} u_k$.

[15%] 5. Let $A \in \mathbb{R}^{n\times n}$.

(a) Prove that A^T and A have the same determinant.

(b) Prove that A^T and A have the same eigenvalues.

(c) Suppose taht A is a Markov matrix. That is, the sum of the entries of any column is 1, and there is at least one non zero entry in every row. Show that 1 is an eigenvalue of A.

[10%] 6. Let $f_i : \mathbb{R}^3 \to \mathbb{R}$, $i = 1, 2, 3$, be given by

$$f_1(x, y, z) = x - 2y,$$
$$f_2(x, y, z) = x + y + z,$$
$$f_3(x, y, z) = y - 3x.$$

(a) Show that f_1, f_2, and f_3 form a basis of the dual space of $\mathbb{R}^3$.

(b) Find the dual basis of $\{f_1, f_2, f_3\}$.

[20%] 7. Let $A \in \mathbb{C}^{n\times n}$ with $A^H = A$, and let $y = (y_1, \ldots, y_n) \in \mathbb{C}^n$ and $z = (z_1, \ldots, z_n) \in \mathbb{C}^n$.

(a) Prove $zAz^H \in \mathbb{R}$.

(b) Prove that every eigenvalue of A is real.

(c) Prove that if y and z are eigenvectors of A corresponding to distinct eigenvalues, then $yz^H = 0$.