國立交通大學 102 學年度碩士班考試入學試題

科目：微積分(4051) 考試日期：102 年 2 月 3 日 第 2 節

系所班別：應用數學系數學建模與科學計算碩士班

【不可使用計算機】＊作答前請先核對試題、答案卷(試卷)與准考證之所組別與考科是否相符！！

1. (10 points) Use the definition to calculate the derivative of $f(x) = \dfrac{x}{x-1}$ at 2, $f'(2) = ?$

2. (10 points) If
$$F(x) = \int_0^x \frac{dt}{1+t^2} + \int_0^{\frac{1}{x}} \frac{dt}{1+t^2},$$
find $F(x)$ and $F'(x)$ explicitly.

3. (15 points) Suppose $f(x)$ is twice-differentiable and one-to-one. Denoting $g(x) = f^{-1}(x)$, show that
$$g''(x) = -\frac{f''(g(x))}{[f'(g(x))]^3}.$$

4. Determine the behavior of the following series: (If the sum is computable; find it!)

   (a) (7 points) $\displaystyle\sum_{n=1}^{\infty}[\sqrt{n+2} - 2\sqrt{n+1} + \sqrt{n}\,];$

   (b) (8 points) $\displaystyle\sum_{n=1}^{\infty} e^n\left(\frac{n}{n+1}\right)^{2n^2}.$

5. (15 points) Among all planes that are tangent to the surface $xy^2z^3 = 1$, find the ones that are farthest from the origin.

6. (15 points) Evaluate the integral $\displaystyle\int_0^2\int_0^3 e^{\max(9x^2,4y^2)}\, dydx$, where
$$\max(9x^2, 4y^2) = \begin{cases} 9x^2 & \text{if } 9x^2 \geq 4y^2; \\ 4y^2 & \text{if } 9x^2 < 4y^2. \end{cases}$$

7. Let f be the function given by
$$\begin{cases} f(x,y) = \frac{xy}{\sqrt{x^2+y^2}} \sin \frac{1}{\sqrt{x^2+y^2}}, & \text{if } (x,y) \neq (0,0) \\ f(0,0) = 0. \end{cases}$$

   (a) (5 points) Find $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$ and justify your answer.

   (b) (5 points) Is f continuous on $\mathbb{R}^2$?

8. (10 points) Evaluate the integral $\displaystyle\iint_D \cos\left(\left(\frac{x}{4}\right)^2 + \left(\frac{y}{5}\right)^2\right) dxdy$, where D is the region enclosed by $\left(\frac{x}{4}\right)^2 + \left(\frac{y}{5}\right)^2 = 1$.