

國立交通大學 102 學年度碩士班考試入學試題

科目：線性代數(4032), 線性代數(4032)

考試日期：102 年 2 月 3 日 第 4 節

系所班別：應用數學系

組別：應數系甲組, 乙組

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【不可使用計算機】*作答前請先核對試題、答案卷(試卷)與准考證之所組別與考科是否相符!!

Note: In the following, I_n denotes the $n \times n$ identity matrix.

1. (15 %) Let $T: \mathbb{R}^3 \rightarrow \mathbb{R}^2$ be defined by

$$T \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x + y + z \\ x + 3y \end{pmatrix}.$$

(a) Find the matrix representation A of T relative to the standard basis

$$\alpha = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\} \text{ for } \mathbb{R}^3, \text{ and}$$

$$\beta = \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\} \text{ for } \mathbb{R}^2. (5 \%)$$

(b) Find the matrix representation B of T relative to the basis

$$\gamma = \left\{ \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\} \text{ for } \mathbb{R}^3, \text{ and}$$

$$\eta = \left\{ \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\} \text{ for } \mathbb{R}^2. (5 \%)$$

(c) Find two nonsingular matrices P and Q such that $A = PBQ$. (5 %)

2. (10 %) Let $A = \begin{pmatrix} 1 & 2 \\ 0 & 1 \\ 1 & 4 \end{pmatrix}$. Find $Q \in M_{3 \times 2}(\mathbb{R})$ and $R \in M_{2 \times 2}(\mathbb{R})$ such that $A = QR$, $Q^T Q = I_2$, and R is an upper triangular matrix.

3. (15 %) Let $A = \begin{pmatrix} 1 & 3 & 0 & -1 & 2 \\ 0 & -2 & 4 & -2 & 0 \\ 3 & 11 & -4 & -1 & 6 \\ 2 & 5 & 3 & -4 & 0 \end{pmatrix}$.

(a) Find a basis of the null space of A . (10 %)

(b) What are the nullity and rank of A ? (5 %)

4. (10 %) Let $A = \begin{pmatrix} -1 & 0 & 0 & 0 \\ -1 & 1 & 1 & -1 \\ -1 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \end{pmatrix}$.

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科目：線性代數(4032), 線性代數(4042)

考試日期：102 年 2 月 3 日 第 4 節

系所班別：應用數學系

組別：應數系甲組, Z 組

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【不可使用計算機】*作答前請先核對試題、答案卷(試卷)與准考證之所組別與考科是否相符!!

- (a) Find all the eigenvalues of A . (5 %)
- (b) Use (a) to find the Jordan canonical form of A . (5 %)
5. (10 %) Let $A, B \in M_{n \times n}(\mathbb{R})$.
- (a) If A is nonsingular and AB is diagonalizable, show that BA is also diagonalizable. (5 %)
- (b) Is BA also diagonalizable if AB is diagonalizable? Justify your answer. (That is : If true, give its proof; otherwise, explain why.) (5 %)
6. (10 %) Show that if u, v are $n \times 1$ column vectors, then
- $$\det(I_n + uv^T) = 1 + v^T u.$$
7. (10 %) Let $A, B \in M_{n \times n}(\mathbb{R})$. Prove the following statements:
- (a) $\text{rank}(A + B) \leq \text{rank}(A) + \text{rank}(B)$. (5 %)
- (b) $\text{rank}(A) = \text{rank}(A^T A)$. (5 %)
8. (10 %) Let V be the subspace of $M_{n \times n}(\mathbb{R})$ consisting of all matrices $A = [a_{ij}]_{i,j=1}^n$ with $\sum_{i=1}^n a_{ii} = 0$. Find the dimension of V and prove your assertion.
9. (10 %) Let V be a complex inner product space with inner product $\langle \cdot, \cdot \rangle$ and its associated norm $\| \cdot \|$, and let T be a linear transformation on V . Prove that if $\|Tx\| = \|x\|$ for all x in V , then $\langle Tx, Ty \rangle = \langle x, y \rangle$ for all x and y in V .