Note: In the following, I_n denotes the $n \times n$ identity matrix.

1. (15 %) Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be defined by

$$T\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x+y+z \\ x+3y \end{pmatrix}.$$

   (a) Find the matrix representation A of T relative to the standard basis

$$\alpha = \{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\} \text{ for } \mathbb{R}^3, \text{ and}$$

$$\beta = \{\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}\} \text{ for } \mathbb{R}^2. \ (5\ \%)$$

   (b) Find the matrix representation B of T relative to the basis

$$\gamma = \{\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\} \text{ for } \mathbb{R}^3, \text{ and}$$

$$\eta = \{\begin{pmatrix} 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix}\} \text{ for } \mathbb{R}^2. \ (5\ \%)$$

   (c) Find two nonsingular matrices P and Q such that $A = PBQ$. (5 %)

2. (10 %) Let $A = \begin{pmatrix} 1 & 2 \\ 0 & 1 \\ 1 & 4 \end{pmatrix}$. Find $Q \in M_{3\times 2}(\mathbb{R})$ and $R \in M_{2\times 2}(\mathbb{R})$ such that $A = QR$, $Q^TQ = I_2$, and R is an upper triangular matrix.

3. (15 %) Let $A = \begin{pmatrix} 1 & 3 & 0 & -1 & 2 \\ 0 & -2 & 4 & -2 & 0 \\ 3 & 11 & -4 & -1 & 6 \\ 2 & 5 & 3 & -4 & 0 \end{pmatrix}$.

   (a) Find a basis of the null space of A. (10 %)

   (b) What are the nullity and rank of A? (5 %)

4. (10 %) Let $A = \begin{pmatrix} -1 & 0 & 0 & 0 \\ -1 & 1 & 1 & -1 \\ -1 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \end{pmatrix}$.

(a) Find all the eigenvalues of A. (5 %)

(b) Use (a) to find the Jordan canonical form of A. (5 %)

5. (10 %) Let $A, B \in M_{n\times n}(\mathbb{R})$.

   (a) If A is nonsingular and AB is diagonalizable, show that BA is also diagonalizable. (5 %)

   (b) Is BA also diagonalizable if AB is diagonalizable? Justify your answer. (That is : If true, give its proof; otherwise, explain why.) (5 %)

6. (10 %) Show that if u, v are $n \times 1$ column vectors, then

$$\det\,(I_n + uv^T) = 1 + v^T u.$$

7. (10 %) Let $A, B \in M_{n\times n}(\mathbb{R})$. Prove the following statements:

   (a) rank $(A+B) \leq$ rank (A) + rank (B). (5 %)

   (b) rank (A) = rank $(A^T A)$. (5 %)

8. (10 %) Let V be the subspace of $M_{n\times n}(\mathbb{R})$ consisting of all matrices $A = [a_{ij}]_{i,j=1}^n$ with $\sum_{i=1}^n a_{ii} = 0$. Find the dimension of V and prove your assertion.

9. (10 %) Let V be a complex inner product space with inner product $\langle\cdot,\cdot\rangle$ and its associated norm $\|\cdot\|$, and let T be a linear transformation on V. Prove that if $\|Tx\| = \|x\|$ for all x in V, then $\langle Tx, Ty\rangle = \langle x, y\rangle$ for all x and y in V.