國立交通大學 102 學年度碩士班考試入學試題

科目：高等微積分(4031) 考試日期：102 年 2 月 3 日 第 3 節

系所班別：應用數學系 組別：應數系甲組

【不可使用計算機】＊作答前請先核對試題、答案卷(試卷)與准考證之所組別與考科是否相符！！

(1) (16 points) Determine the convergency (absolutely convergent, conditionally convergent or divergent) of the following series.

$$\text{(a) (8 points)} \sum_{n=1}^{\infty}(-1)^{n+1}\frac{\log(n+1)}{n}, \quad \text{(b) (8 points)} \sum_{n=1}^{\infty}(n^{1/n}-1).$$

(2) (20 points) Let M, N be sets and d_M, d_N be metrics on M, N and $A \subset M$. Suppose A is compact and $f : A \to N$ is continuous. Prove that

(a) (10 points) $f(A)$ is compact;

(b) (10 points) f is uniformly continuous on A.

(3) (14 points) Let $f : (a, b) \to \mathbb{R}$ be a differentiable function.

(a) (10 points) Prove that if $f' > 0$ on (a, b), then $f(x) < f(y)$ for $x, y \in (a, b)$ and $x < y$.

(b) (4 points) Is the converse of (a) true? Prove or disprove it.

(4) (20 points) Let

$$f(x) = \begin{cases} x + 2x^2 \sin\dfrac{1}{x}, & x \neq 0, \\ 0 & x = 0. \end{cases}$$

(a) (6 points) Find $f'(0)$?

(b) (8 points) Is f locally invertible near 0? Justify your answer.

(c) (6 points) Does this result contradict the inverse function theorem? Why?

(5) (16 points) Let

$$f_n(x) = \begin{cases} 0, & \text{if } x < \dfrac{1}{n+1}, \\ \sin^2\dfrac{\pi}{x}, & \text{if } \dfrac{1}{n+1} \leq x \leq \dfrac{1}{n}, \\ 0, & \text{if } x > \dfrac{1}{n}. \end{cases}$$

(a) (8 points) Show that (f_n) converges to a continuous function.

(b) (8 points) Does (f_n) converge uniformly?

(6) (14 points)

(a) (8 points) Let f be a positive continuous function on $[0, 1]$ with maximum value M. Prove that

$$\lim_{n\to\infty}\left(\int_0^1 |f(x)|^n \, dx\right)^{1/n} = M.$$

(b) (6 points) If f is a continuous function on $[0, 1]$ with maximum value M and minimum value m. Evaluate

$$\lim_{n\to\infty}\left(\int_0^1 |f(x)|^n \, dx\right)^{1/n}$$

in terms of M and m.