題號： 419
科目：工程數學(D)
節次： 6

國立臺灣大學 102 學年度碩士班招生考試試題

題號： 419
共 4 頁之第 1 頁

第一大題：(50%)

本大題共十小題單選題，考生請作答於電腦閱卷『答案卡』。每小題有 5 個選項，答對得 5 分，答錯或劃記多於一個選項者倒扣 1 分，倒扣到本大題的分數為零分為止。未作答者，不給分亦不扣分。

1. (5%) Let $x, y \in \mathbb{N}$ be two natural numbers. For $n \in \mathbb{N}$, let E_n be the set of pairs of natural numbers (x, y) where n divides both x and y. What is the probability $P[E_n]$ that any randomly chosen natural number is in E_n?

   (A) $1/n$.
   (B) $1/n^2$.
   (C) $1/n!$.
   (D) $1/\log n$.
   (E) None of the above.

2. (5%) (Continued from 1.) Given two randomly chosen natural numbers $x, y \in \mathbb{N}$, denote Q as the probability that x and y are relatively prime (i.e. no common positive divisors except 1). Which of the following range does Q lie in?

   (A) $[0.0, 0.3)$.
   (B) $[0.3, 0.5)$.
   (C) $[0.5, 0.7)$.
   (D) $[0.7, 0.9)$.
   (E) $[0.9, 1.0]$.

3. (5%) Consider an experiment of repeated throwing a fair die. Starting from the first throw, if the die comes up as 6, then the experiment ends (duration of the experiment=0). If, on the other hand, the die comes up as k, $\forall 1 \le k \le 5$, then the next throw should wait for k minutes to start. Let T be the expected duration of the experiment rounded to the nearest integer (minutes). What is the sum of all digits in T? (For example, if $T = 21$, then the sum is 3.)

   (A) 3.
   (B) 6.
   (C) 9.
   (D) 10.
   (E) None of the above.

4. (5%) Five people get on an elevator on the first floor of an 11-story building. Assume that each person selects a destination floor from 10 possible floors (i.e. from 2 to 11) uniformly at random and independently of others. Also, assume that no new people get on the elevator above the first floor. Let S be the expected number of stops this elevator makes above the first floor, and Z be the sum of all digits (including all decimal places) in S. Then which of the following statements is true?

   (A) $15 < Z \le 19$.
   (B) $19 < Z \le 22$.
   (C) $22 < Z \le 24$.
   (D) $24 < Z \le 25$.
   (E) None of the above.

見背面

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5. (5%) Assume that the number of typhoons in a given year follows the Poisson distribution with a mean value of 3, and that the occurrence of typhoons in any given year is independent of that in any other years. It can be derived that the probability of having at least 3 typhoons in the 2-year period from 2011 to 2012 can be written as $A + B * e^C$ with A, B, C being constant numbers. What is the value of $A + B + C$?

  (A) 15.
  (B) −20.
  (C) 35.
  (D) −30.
  (E) None of the above.

6. (5%) Let X be the amount of a claim (in 10 thousands) made by a home damaged by the typhoon. Assume that X has the following probability density distribution:
$$f_X(x) = \begin{cases} 3x^{-4}, & x > 1, \\ 0, & \text{otherwise.} \end{cases}$$

Suppose 3 such claims are made during a typhoon attack and claim amounts for individual homes are independent of each other. Let Y be largest of the three claims (in 10 thousands). Then which of the following statements is true?

  (A) The probability that Y is smaller than 1 is 0.2.
  (B) The probability that Y is smaller than 2 is larger than 0.5.
  (C) The probability that Y is smaller than 2 is smaller than 0.6.
  (D) The probability that Y is smaller than 3 is larger than 0.9.
  (E) None of the above.

7. (5%) (Continued from 6.) Which of the following number is closest to the expected value $E[Y]$ of Y?

  (A) 1.8.
  (B) 2.5.
  (C) 3.2.
  (D) 4.1.
  (E) 5.4.

8. (5%) Let the moment of a discrete random variable X be given by $E[X^k] = 0.8$, $\forall k = 1, 2, \dots$. Let $R = P[X = 0]$ be the probability that X equals 0. Then which of the following statements is true?

  (A) $0 \le R \le 0.2$.
  (B) $0.2 < R \le 0.4$.
  (C) $0.4 < R \le 0.6$.
  (D) $0.6 < R \le 0.8$.
  (E) $R > 0.8$.

9. (5%) X is a three-dimensional Gaussian random vector with expected value $\mu_{\mathbf{X}} = [4 \quad 8 \quad 6]'$ and covariance matrix

$$\mathbf{C_X} = \begin{bmatrix} 4 & -2 & 1 \\ -2 & 4 & -2 \\ 1 & -2 & 4 \end{bmatrix}.$$

Let D be the maximum number in the correlation matrix $\mathbf{R_X}$. Then which of the following statements is true?

(A) $20 \leq D < 30$.

(B) $30 \leq D < 40$.

(C) $40 \leq D < 50$.

(D) $50 \leq D < 60$.

(E) None of the above.

10. (5%) (Continued from 9.) Let random vector $\mathbf{Y} = \mathbf{A}\mathbf{X} + \mathbf{b}$, where

$$\mathbf{A} = \begin{bmatrix} 1 & 1/2 & 2/3 \\ 1 & -1/2 & 2/3 \end{bmatrix}$$

and $\mathbf{b} = [-4 \quad -4]'$. Let $\mathbf{C_Y}$ be the covariance matrix of $\mathbf{Y}$. Then which of the following statements is true?

(A) $\mathbf{C_Y}$ is a 3×3 matrix.

(B) The largest number in $\mathbf{C_Y}$ is smaller than 12.

(C) The smallest number in $\mathbf{C_Y}$ is larger than 5.

(D) The sum of all numbers in $\mathbf{C_Y}$ is larger than 30.

(E) None of the above.

見背面

第二大題：(50%)

11. (30%) Label the following statements as being true or false. *(No explanation is needed. Each correct answer gets 3% and each wrong answer gets 0%)*:

   (a) *Every* nonzero $n \times n$ skew-symmetric matrix is not diagonalizable.

   (b) Let A and B be $n \times k$ and $k \times m$ matrices respectively. If the columns of A are linearly independent and the columns of B are linearly independent, then the columns of AB are also linearly independent.

   (c) Let A be $n \times k$ and $\mathbf{b} \in \mathcal{R}^n$. There exists a *unique* $\mathbf{v}$ in $\mathcal{R}^k$ such that $\|A\mathbf{v} - \mathbf{b}\| < \|A\mathbf{x} - \mathbf{b}\|$ for any $k \times 1$ vector $\mathbf{x} \neq \mathbf{v}$.

   (d) Let V be a subspace of $\mathcal{R}^n$. Then $(V^\perp)^\perp = V$.

   (e) Let $T : \mathcal{R}^5 \to \mathcal{R}^4$ be linear. Then T is not one-to-one.

   (f) Let $A = [\mathbf{a}_1\ \mathbf{a}_2\ \dots\ \mathbf{a}_n]$ be a square matrix and $\mathbf{b}$ be a linear combination of $\mathbf{a}_2, \mathbf{a}_3, \dots, \mathbf{a}_n$. Then $\det A = \det[\mathbf{a}_1 + \mathbf{b}\ \ \mathbf{a}_2\ \mathbf{a}_3\ \dots\ \mathbf{a}_n]$.

   (g) Similar matrices have the same reduced row echelon form.

   (h) There exists a non-invertible 3×3 matrix A with characteristic polynomial $f(t) = t^3 + 2t^2 - 3t$.

   (i) Every vector space is an inner product space.

   (j) The set of all diagonalizable $n \times n$ matrices is a subspace of $\mathcal{M}_{n\times n}$.

12. (10%) Let the reduced row echelon form of $A = [\mathbf{a}_1\ \mathbf{a}_2\ \mathbf{a}_3\ \mathbf{a}_4\ \mathbf{a}_5]$ be given by

$$\begin{bmatrix} 1 & 0 & 2 & 0 & 1 \\ 0 & 1 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

   Find the reduced row echelon form of B, where $B = [\mathbf{a}_2\ \mathbf{a}_1\ \mathbf{a}_5\ \mathbf{a}_4\ \mathbf{a}_3]$. *(No explanation is needed. Each correct column gets 2% and each wrong column gets 0%)*

13. (10%) Let T be a linear operator on $\mathcal{M}_{2\times 2}$ and $T(A) = A - 2A^T$. Find an expression for

$$T^{-1}\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right).$$

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試題隨卷繳回

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