國立政治大學 102 學年度碩士班招生考試試題

考試科目	線性代數	所別	應用數學系	考試時間	2月24日(日)第 2 節

Please show all your work.

1. Let T be a linear operator on a vector space V, let v be a nonzero vector in V, and let W be the T-cyclic subspace of V generated by v. Prove that

   (a) **(5%)** W is T-invariant.
   (b) **(10%)** Any T-invariant subspace of V containing v also contains W.

2. Let $C[-1,1]$ denote the inner product space of real continuous functions defined on $[-1,1]$. For any $f,g \in C[-1.1]$, the inner product is defined as

$$<f,g>=\int_{-1}^{1} f(x)g(x)dx$$

   (a) **(5%)** Show that $u_1(x)=\frac{1}{\sqrt{2}}$ and $u_2(x)=\frac{\sqrt{6}}{2}x$ form an orthonormal set of vectors.

   (b) **(10%)** Let W be the subspace of $C[-1,1]$ spanned by $u_1(x)$ and $u_2(x)$. Find $w(x)\in W$ such that $w(x)$ minimizes $\|w(x)-h(x)\|$ where $\|\cdot\|$ is the norm induced by the inner product and $h(x)=x^{1/3}+x^{2/3}$.

3. **(10%)** Let A be an $n\times n$ matrix and let $B=I-2A+A^2$. Show that if $\lambda=1$ is an eigenvalue of A, then the matrix B will be singular.

4. Let Q be a 3×3 orthogonal matrix whose determinant is equal to 1.

   (a) **(10%)** If the eigenvalues of Q are all real and if they are ordered so that $\lambda_1 \geq \lambda_2 \geq \lambda_3$, determine the values of all possible triples of eigenvalues $(\lambda_1,\lambda_2,\lambda_3)$.

   (b) **(10%)** In the case that the eigenvalues λ_2 and λ_3 are complex, what are the possible values for λ_1? Explain.

備註	試題隨卷繳交

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5. **(10%)** Let $A=[a_{i,j}]$ be an $n\times n$ matrix with eigenvalues $\lambda_1,\ldots,\lambda_n$. Show that

$$\lambda_j = a_{j,j} + \sum_{i\neq j}(a_{i,i}-\lambda_i) \text{ for } j=1,\ldots,n.$$

6. A linear operator T on a finite-dimensional inner product space with inner product $\langle\bullet,\bullet\rangle$ is called **positive definite** if T is self-adjoint and $\langle T(x),x\rangle>0$ for all $x\neq 0$.

   (a) **(15%)** Let V be a finite-dimensional inner product space with inner product $\langle\bullet,\bullet\rangle$, and let T be a positive definite linear operator on V. Prove that $\langle x,y\rangle'=\langle T(x),y\rangle$ for all x and y in V defines another inner product on V.

   (b) **(15%)** Prove the converse of (a): Let V be a finite-dimensional inner product space with inner product $\langle\bullet,\bullet\rangle$ and let $\langle\bullet,\bullet\rangle'$ be any other inner product on V. Then, there exists a unique linear operator T on V such that $\langle x,y\rangle'=\langle T(x),y\rangle$ for all x and y in V.

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