(20 points for each problem)

1. (a) As in fig. 1, an object is shot up at an initial velocity $v_0 = \sqrt{GM/R}$ at an angle 60^0 relative to the vertical line, where M is the Earth's mass. What is the largest distance r of the object to the Earth's center ? (b) We want to dispose of nuclear waste by crashing it into the Sun (mass M). Given the radius of Earth's orbit r_E and Sun's radius R_s, the most efficient way is to shoot the waste m at a velocity Δv relative to the Earth in backward direction so that the circular orbit of the waste is turned into an elliptical orbit with R_s as its perihelion distance. What is the Δv required ? (10+10 points)

2. A circular ring is suspended in a horizontal plane by three strings, each of length ℓ, which are attached symmetrically to the ring and are connected to fixed points lying in a plane above the ring. At equilibrium, each string is vertical. Write down the eq. of motion for small angle θ to show that the angular frequency of small rotational oscillations about the vertical through the center of the ring is $\omega = \sqrt{g/\ell}$.

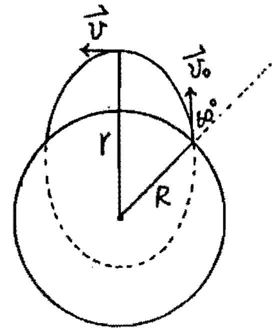
3. (a) By computing $dL(q_i(t),\dot{q}_i(t),t)/dt$, show that the Hamiltonian H is conserved if the Lagrangian L has no explicit time dependence. (b) A simple pendulum consists of a mass m attached to a string of length ℓ. After the pendulum is set into motion, ℓ is shortened at a constant rate, i.e. $\ell(t) = \ell_0 - \alpha t$ with $\alpha = const$. The suspension point remains fixed. L=? H=? Is H conserved ? (10+10 points)

4. An object rotates about a fixed point at instantaneous angular velocity $\vec{\omega} = (\omega_1, \omega_2, \omega_3)$. Show that the kinetic energy T and angular momentum $\vec{L}$ are related to $\vec{\omega}$ by $T = \frac{1}{2}\sum_{i=1}^{3}\sum_{j=1}^{3} I_{ij}\omega_i\omega_j$ and $L_i = \sum_{j=1}^{3} I_{ij}\omega_j$,

where $I_{ij} \equiv \sum_{\substack{\alpha \\ (mass\ cell)}} m_\alpha \left(\delta_{ij} r_\alpha^2 - x_{\alpha i} x_{\alpha j}\right)$, $r_\alpha^2 \equiv \sum_{k=1}^{3} x_{\alpha k}^2$ for the α-th mass cell.

(You might need these 2 fomulas: $(\vec{A}\times\vec{B})^2 = A^2B^2 - (\vec{A}\cdot\vec{B})^2$, $\vec{A}\times(\vec{B}\times\vec{C}) = \vec{B}(\vec{A}\cdot\vec{C}) - \vec{C}(\vec{A}\cdot\vec{B})$.)

5. The solution of the wave eq. for a string of linear density ρ, tension τ, length L and fixed at both ends are given by $q(x,t) = \sum_{n=1}^{\infty} \sin(n\pi x/L)[\alpha_n \cos(n\omega t) - \beta_n \sin(n\omega t)]$, where $\omega = \frac{\pi}{L}\sqrt{\frac{\tau}{\rho}}$ is the fundamental angular frequency. As in fig. 5, the center of the string is displaced a distance h and then released at $t = 0$ from rest. Find $q(x,t)$ for all time. (You might need: $\int x\sin(ax)dx = \frac{1}{a^2}\sin(ax) - \frac{x}{a}\cos(ax)$.)



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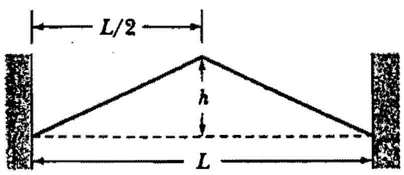

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