Work out all problems and no credit will be given for an answer without reasoning.

1. (a) (8%) Let B be a subset of a vector space V. Show that B is a basis for V if and only if every member of V is a unique linear combination of the elements of B.

   (b) (4%) Let T be a linear transformation of a vector space V. Prove that the set $\{\mathbf{v} \in V \mid T(\mathbf{v}) = 0\}$, the **kernel** of T, is a subspace of V.

   (c) (8%) Let V and W are vector spaces over a field F. Define a vector space isomorphism from V to W is a one-to-one linear transformation from V onto W. If V is a vector space over F of dimension n, prove that V is isomorphic as a vector space to $F^n = \{(a_1, a_2, \ldots, a_n) \mid a_i \in F\}$.

2. (a) (8%) Let

$$A = \begin{bmatrix} -3 & 5 & -20 \\ 2 & 0 & 8 \\ 2 & 1 & 7 \end{bmatrix}$$

   Find an invertible matrix P such that $P^{-1}AP$ is a diagonal matrix.

   (b) Let A and C be $n \times n$ matrices, and let C be an invertible.

   i. (4%) Show that the eigenvalues of A and of $C^{-1}AC$ are the same.

   ii. (8%) Prove that, if $\mathbf{v}$ is an eigenvector of A with corresponding eigenvalue λ, then $C^{-1}\mathbf{v}$ is an eigenvector of $C^{-1}AC$ with corresponding eigenvalue λ. Then prove that all eigenvectors of $C^{-1}AC$ are of the form $C^{-1}\mathbf{v}$, where $\mathbf{v}$ is an eigenvector of A.

3. (a) (10%) Find an orthonormal basis for the subspace spanned by the set $\{1, x, x^2\}$ of the vector space $C_{[-1,1]}$ of continuous functions with domain $-1 \le x \le 1$, where the inner product is defined by $< f, g > = \int_{-1}^{1} f(x)g(x)\, dx$.

   (b) (5%) Subspaces U and W of $\mathbb{R}^n$ are **orthogonal** if $\mathbf{u} \cdot \mathbf{w} = 0$ for all $\mathbf{u}$ in U and all $\mathbf{w}$ in W. Let U and W be orthogonal subspaces of $\mathbb{R}^n$, and let $dim(U) = n - dim(W)$. Prove that each subspace is the orthogonal complement of the other.

4. (a) (8%) Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be a linear transformation defined by

$$T((x_1, x_2, x_3)) = (x_1 + x_3, x_2, x_1 + x_3).$$

   Find the eigenvalues λ_i and the corresponding eigenspaces of T. Determine whether the linear transformation T is diagonalizable.

   (b) (7%) Let $U = [u_{ij}]$ be a square matrix with complex entries. Define the matrix U is **unitary** if $U^*U = I$, where $U^* = [\overline{u_{ij}}]^T$. Prove that the product of two $n \times n$ unitary matrices is also a unitary matrix. What about the sum of two $n \times n$ unitary matrices?

(背面仍有題目,請繼續作答)

5. (a) (9%) Find a Jordan canonical form and a Jordan basis of

$$A = \begin{bmatrix} 4 & 0 & 0 \\ 2 & 1 & 3 \\ 5 & 0 & 4 \end{bmatrix}$$

(b) (6%) Let

$$A = \begin{bmatrix} 1 & 1 & -3 \\ 0 & 1 & 1 \\ 3 & -1 & 1 \end{bmatrix} \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} -13 \\ 6 \\ -7 \end{bmatrix}$$

Find a permutation matrix P, a lower-triangular matrix L, and an upper-triangular matrix U such that $PA = LU$. Then solve the system $A\mathbf{x} = \mathbf{b}$, using P, L, and U.

6. (15%) Let V be a finite-dimensional complex or real vector space with inner product $< \cdot, \cdot >$ and suppose that W is a subspace of V. Let

$$W^{\perp} = \{\mathbf{v} \in V \mid \ < \mathbf{v}, \mathbf{w} >= 0 \text{ for every } \mathbf{w} \in W\}.$$

Show that $W^{\perp}$ is a subspace of V and

$$V = W \oplus W^{\perp},$$

that is each $\mathbf{v} \in V$ can be written uniquely as a sum $\mathbf{v} = \mathbf{v}_1 + \mathbf{v}_2$ where $\mathbf{v}_1 \in W$ and $\mathbf{v}_2 \in W^{\perp}$.