Part I.
Linear Algebra (50%)

1. Given $x_1 = (1, 1, 1)^T$ and $x_2 = (3, -1, 4)^T$: (a) Do x_1 and x_2 span $\mathbb{R}^3$? Explain. (3%) (b) Let x_3 be a third vector in $\mathbb{R}^3$ and set $X = (x_1, x_2, x_3)$. What condition(s) would X have to satisfy in order for x_1, x_2, and x_3 to form a basis for $\mathbb{R}^3$? (3%) (c) Find a third vector x_3 that will extend the set $\{x_1, x_2\}$ to a basis for $\mathbb{R}^3$. (4%)

2. Let $A=\begin{bmatrix} 3 & -1 & -2 \\ 2 & 0 & -2 \\ 2 & -1 & -1 \end{bmatrix}$, (a) Find a basis and rank for the row space of A (4%); (b) Find a diagonal matrix D of A and calculate A^{10} (8%); (c) Compute e^A, the exponential of A (4%). (d) Let L be the linear operator mapping $\mathbb{R}^3$ into $\mathbb{R}^3$ defined by $L(x)=Ax$ and let $v_1=\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, $v_2=\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$, $v_3=\begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix}$. Find the transition matrix V corresponding to a change of basis from $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ to $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$, and use it to determine the matrix B representing L with respect to $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$. (8%)

3. Let $A=\begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 1 & 3 & 4 \end{bmatrix}$, find the bases for the (a) null space of A, N(A), (b) range of A, R(A), (c) $N(A^T)$, and $R(A^T)$. (16%)

Part II.

Discrete Mathematics (50%)

一、單選題

1. (5%) For an alphabet Σ, let $A, B, C \subseteq \Sigma^*$. Which statement is **FALSE**?
   (A). $(AB)C = A(BC)$.
   (B). $AB \cup AC = A(B \cup C)$.
   (C). $AB \cap AC \subseteq A(B \cap C)$.
   (D). $(A \cup B)^* = (A^* B^*)^*$.
   (E). $A^* A^* = A^*$.

2. (5%) Which statement is **FALSE**?
   (A). If A={1, 2, 3, …, 10}, the number of functions f: A → A satisfy $f^{-1}(\{1, 2, 3\})=\emptyset$, $f^{-1}(\{4, 5\})=\{1, 3, 7\}$, and $f^{-1}(\{8, 10\})=\{8, 10\}$ is 7776.
   (B). The number of nonnegative integer solutions of $x_1+ x_2+ x_3+\ldots+ x_7=37$ and $x_1+ x_2+ x_3=6$ where $x_1, x_2, x_3 > 0$ is $10 * \binom{34}{31}$.
   (C). The coefficient of x^5 in $f(x) = (1 - 2x)^{-7}$ is $32 * \binom{11}{5}$.
   (D). The coefficient of x^2yz^2 in the expansion of $[(x/2) + y - 3z]^5$ is $\frac{145}{2}$.

3. (5%) Which statement is **TRUE**?
   (A). Let (A, **R**) be a poset. If (A, **R**) is a lattice, then it is a total order.
   (B). If A={1, 2, 3, 4, 5, 6, 7} and the relation R is defined as if $(x, y) \in$ R, x-y is a multiple of 3, then R is an equivalence relation of A.
   (C). The subset relation is a total ordered relation.
   (D). If (A, **R**) is a poset and $B \subseteq A$, then B has more than one lub.

4. (5%) Define the connective "Nand" by $(p \uparrow q) \Leftrightarrow \neg(p \wedge q)$, for any statements p, q. Which statement is **FALSE**?
   (A). $\neg p \Leftrightarrow (p \uparrow p)$.
   (B). $(p \vee q) \Leftrightarrow (p \uparrow p) \uparrow (q \uparrow q)$.
   (C). $(p \wedge q) \Leftrightarrow (p \uparrow q) \uparrow (p \uparrow q)$.
   (D). $p \rightarrow q \Leftrightarrow p \uparrow (p \uparrow q)$.

5. (5%) Let f: X → Y and g: Y → Z be functions. Which statement is **FALSE**?
   (A). If $g \circ f$ is one-to-one, f is one-to-one.
   (B). If $g \circ f$ is one-to-one, g is one-to-one.
   (C). If $g \circ f$ is onto, g is onto.
   (D). If $g \circ f$ is bijection, g is onto.
   (E). If $g \circ f$ is bijection, f is one-to-one.

二、 計算題

1. (15%) Use the **recurrence relation** to determine the number of n-digit quaternary (0, 1, 2, 3) sequences in which there is never a '0' anywhere to the right of a '3'.

2. (10%) Let A={1, 2, 3, 4, 5} and B={u, v, w, x, y}. Determine the number of one-to-one functions f : A→B where $f(1) \neq$ v, w, $f(2) \neq$ u, w, $f(3) \neq$ x, y and $f(4) \neq$ v, x, y.