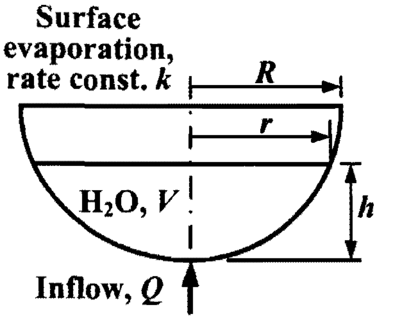
Problem 1 (15 points)



As sketched above, a hemispherical tank is to be filled with water through an inlet in its bottom. Suppose that the radius of the tank is R, and that water is pumped in at a volumetric flowrate of Q. Also, as the tank fills, it loses water through evaporation at a volumetric rate that is proportional to the water surface area A with a proportionality constant k (i.e., the evaporation rate is kA). Note that $R^2 = (R-h)^2 + r^2$, and so we can write $A = \pi r^2 = \pi h(2R-h)$. Moreover, the volume of water shown in the sketch is $V = \pi h^2(R - h/3)$.

(a) Show that

$$\pi h(2R-h)\frac{dh}{dt} = Q - \pi k h(2R-h).$$

(b) Consider now the special case that $Q = k \cdot \pi R^2$, and solve the resulting differential equation. Assume that the tank initially is empty.

Problem 2 (10 points)

(a) Given that $y = \sin x$ is a solution of

$$\frac{d^4y}{dx^4} + 2\frac{d^3y}{dx^3} + 11\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 10y = 0,$$

find the general solution of the differential equation.

(b) Find a linear second-order differential equation with constant coefficients for which $y_1 = 1$ and $y_2 = e^{-x}$ are solutions of the associated homogeneous equation and $y_p = x^2/2 - x$ is a particular solution of the nonhomogeneous equation.

Problem 3 (15 points)

Use *Laplace transform* to solve the following initial-value problem:

$$\frac{d^2x}{dt^2} + 3\frac{dy}{dt} + 3y = 0, \quad \frac{d^2x}{dt^2} + 3y = te^{-t},$$

with $x(0) = y(0) = 0$ and $\frac{dx}{dt}(0) = 2$.

Problem 4 (15 points)

In his book *Liber Abbaci*, published in 1202, Leonardo Fibonacci of Pisa speculated on the reproduction of rabbits:

How many pairs of rabbits will be produced in a year beginning with a single pair, if every month each pair bears a new pair which become productive from the second month on?

The answer to his question is contained in a sequence known as a Fibonacci sequence, which can be defined recursively by a second-order difference equation

$$x_n = x_{n-2} + x_{n-1}, \quad n = 2, 3, \ldots,$$

with $x_0 = 1$ and $x_1 = 1$. To understand this equation, think of x_{n-2} as the number of adult (productive) pairs after the $(n-2)$-th months. As each adult pair bears a new pair one month later, there are x_{n-2} baby pairs—and x_{n-1} adult pairs by definition—after the $(n-1)$-th month. One more month later, i.e., after the n-th month, such baby pairs become adults as well, and so there are $x_{n-2} + x_{n-1}$ adult pairs in total, which equals x_n by definition; hence the above equation.

(a) Now, if we let $y_{n-1} = x_{n-2}$, then $y_n = x_{n-1}$, and the difference equation above can be written as a system of first-order difference equations

$$x_n = x_{n-1} + y_{n-1}, \quad y_n = x_{n-1}.$$

Write this system in matrix form $\mathbf{X}_n = \mathbf{A}\mathbf{X}_{n-1}$, $n = 2, 3, \ldots$, where $\mathbf{X}_n = (x_n, y_n)^T$ and $\mathbf{A}$ is a 2×2 coefficient matrix.

(b) Show that

$$\mathbf{A}^m = \frac{1}{\lambda_2 - \lambda_1} \begin{pmatrix} \lambda_2^{m+1} - \lambda_1^{m+1} & \lambda_2^m - \lambda_1^m \\ \lambda_2^m - \lambda_1^m & \lambda_2^{m-1} - \lambda_1^{m-1} \end{pmatrix},$$

where $\lambda_{1,2}$ are the distinct eigenvalues of $\mathbf{A}$.

(c) Use the result in part (a) to show $\mathbf{X}_n = \mathbf{A}^{n-1}\mathbf{X}_1$.

(d) Use the results in parts (b) and (c) to find the number of adult pairs, baby pairs, and total pairs of rabbits after the twelfth month.

Problem 5 (10 points)

The electric field at a point $P(x, y, z)$ due to a point charge q located at the origin is given by the inverse square field

$$\mathbf{E} = q\frac{\mathbf{r}}{|\mathbf{r}|^3},$$

where $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ and $|\mathbf{r}| = \sqrt{x^2 + y^2 + z^2}$.

(a) Suppose S is a closed surface, S_a is a sphere $x^2 + y^2 + z^2 = a^2$ lying completely within S, and D is the region bounded by S and S_a. Show that the outward flux of $\mathbf{E}$ for the region D is zero.

(b) Use the result of part (a) to prove Gauss' law:

$$\iint_S (\mathbf{E} \cdot \mathbf{n})\, dS = 4\pi q,$$

that is, the outward flux of the electric field $\mathbf{E}$ through *any* closed surface containing the origin is $4\pi q$.

Problem 6 (25 points)

Solve the boundary-value problem

$$\begin{aligned}
&k\frac{\partial^2 u}{\partial x^2} - hu = \frac{\partial u}{\partial t}, \quad 0 < x < \pi,\ t > 0,\\
&u(0, t) = 0, \quad u(\pi, t) = u_0, \quad t > 0,\\
&u(x, 0) = 0, \quad 0 < x < \pi.
\end{aligned}$$

The above partial differential equation is a form of the heat equation when heat is lost by convection from the lateral surface of a thin rod into a medium at zero temperature.

Problem 7 (10 points)

Evaluate

$$\int_C \bar{z}\, dz,$$

where C is given by $x = 3t$, $y = t^2$, $-1 \le t \le 4$.