1. (20%) Consider the following problem:

(P-Primal) Maximize: $Z = 2x_1 + 3x_2 + 6x_3$

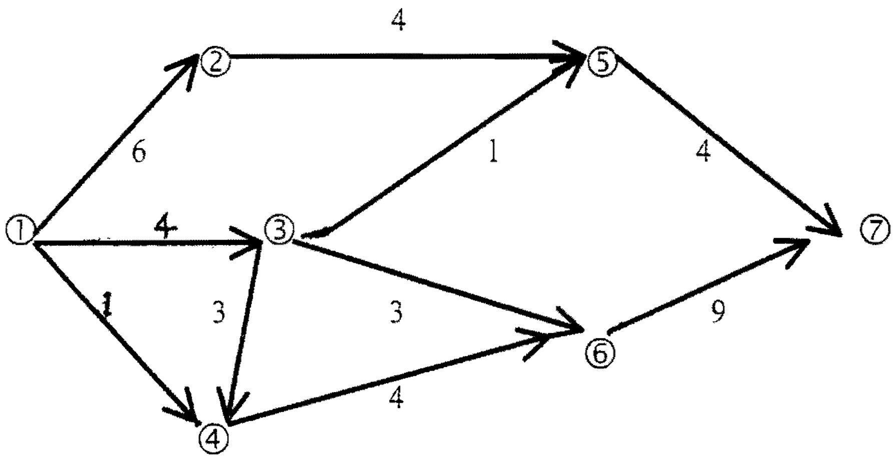
s.t. $x_1 + 2x_2 + 3x_3 \le 10$

$x_1 - 2x_2 + 2x_3 \le 6$

$x_1, x_2, x_3 \ge 0$

Write the dual problem (D) and solve P by the simplex method. At each iteration, identify dual variables and show which dual constraints are violated. Also verify that at termination, feasible solutions of both problems are at hand, with equal objectives, and with complementary slackness conditions.

2. (15%) For the following network where node 1 is the source and node 7 is the sink, and the value on each arc is the capacity of the arc, formulate the maximum flow problem as a linear progamming model. Use Augmenting path algorithm to find the maximum flow. Clearly show the computational details. Illustrate the Maximum flow minimum cut theorem by indicating the location of such cut(s).




3. (15%) A mutual fund of \$100 millions will be invested in three stocks. Let $S_i$ be the random variable representing the annual return on \$1 invested in stock $i$, $i$=1, 2, 3. We are given the following information (expected value $E$, variance $V$, and covariance $Cov$):

$E(S_1) = 0.14, V(S_1) = 0.2, E(S_2) = 0.11, V(S_2) = 0.08, E(S_3) = 0.1, V(S_3) = 0.18,$

$Cov(S_1, S_2) = 0.05, Cov(S_1, S_3) = 0.02, Cov(S_2, S_3) = 0.03$

Formulate this investment problem as a non-linear programming model to find the portfolio that attains an expected annual return of at least 12% and minimizes the variance of the annual dollar return on the portfolio. Clearly define decision variables, objective function, and constraints.

**4** (30 %) In many queueing systems, customers arrive to the system in groups but are still served individually. We say that such a queueing system has bulk arrivals. The system is no longer a birth-death process since customers don't arrive one at a time, but can still be analyzed using tools we have learned.

Consider a small emergency room that has 2 operating tables and 2 beds for holding waiting patients. The interarrival times of ambulances follow an exponential distribution with mean of 30 minutes. A given ambulance may bring into the emergency room 1 or 2 patients with probabilities 60% and 40% respectively. Patients who arrive and find no bed to wait in are taken to a neighboring hospital (i.e. if an ambulance arrives and there are not enough beds for everyone in the ambulance, the ambulance drives to the next hospital without dropping anyone off). A patient's length of stay on an operating table follows an exponential distribution with mean 2.5 hours. Operation durations are independent of each other and of the arrival process.

a) (8 points) Model this problem as a CTMC so that you are capable of answering the following questions. In particular:
   - What is the stochastic process Y(t)?
   - What does the time index t represent?
   - What is the state space and what do the states represent?
   - What is the transition rate matrix G?

b) (3 points) Suppose there is currently one patient in the emergency room. Records show that patient arrived 90 minutes ago and immediately went into surgery. What is the expected remaining time until that patient comes out of surgery (leaves the operating room)? [Numerical Answer]

c) (4 points) If there is only one patient in the hospital, what is the probability that patient is discharged (leaves) before an ambulance arrives? [Numerical Answer]

d) (6 points) Suppose the emergency room is currently empty. The hospital administrator has asked to be notified when the emergency room becomes completely full. Describe how you would calculate the expected amount of time until that occurs. [No numerical answer required]

e) (3 points) Write out (but do not solve) the system of equations to solve for the long-run percentage of time spent in each state.

f) (6 points) How would you calculate the expected number of lost patients (i.e. those who do not enter the emergency room) in one hour? [No numerical answer required. Also note that I want the number of lost *patients* not lost *ambulances*]

**5** (20 %) Orders arrive to a toy factory according to a Poisson process with a rate of 8 per day and are immediately sent to the toy-building station. There are five toy builders and the time it takes each to build a toy follows an exponential distribution with a mean of 8 hours. Once a toy is built, it is sent to be painted. There are four painters and each can paint 2.5 toys per day on average (painting times are also from an exponential distribution). After being painted, the toy is sent to the inspection station. There is one toy inspector and her mean inspection time (exponentially distributed) is 2 hours. Toys are defective with probability *p*. When a toy is defective, it is scrapped and a replacement order is issued to the toy-building station. Otherwise, the toy is shipped.

a) (6 points) Solve for the total arrival rates to each station ($\lambda^{(j)}$) in terms of *p*.
b) (5 points) For what values of *p* is the entire system stable?
c) (9 points) Suppose that $p = 1/9$. It has been calculated that the average length of the queue at each station are as follows:

| | |
|---|---|
| Station 1: Building | $L_q^{(1)} = 0.3542$ |
| Station 2: Painting | $L_q^{(2)} = 7.0898$ |
| Station 3: Inspection | $L_q^{(3)} = 2.2500$ |

Calculate the average total amount of time it takes from the initial order until the toy is shipped.