# 國立臺灣大學 113 學年度碩士班招生考試試題



1. Let $U$ be any random variable and $V$ be any other nonnegative random variable. Denote the cdf of $U+V$ and $U$ by $F_{U+V}$ and $F_U$ respectively. Show that

$$F_{U+V}(t) \leq F_U(t) \text{ for every } t.$$

(8 points)

2. Let $X_1, \ldots, X_n$ be a random sample from the pdf $f(x \mid \theta, \nu) = \frac{\theta \nu^\theta}{x^{\theta+1}}$, where $\theta > 0$ and $x \geq \nu > 0$. Find the MLE for both $\theta$ and $\nu$. (10 points)

3. Suppose an experimenter collects $n$ i.i.d. samples $(A_i, B_i)$, $i = 1, \ldots, n$, where $A_i$'s and $B_i$'s are binary random variables. Given the marginal counts of $A_i$'s and $B_i$'s, we have the following $2 \times 2$ contingency table

| | $A=0$ | $A=1$ | Row Total |
|---|---|---|---|
| $B=0$ | $X$ | $n_{1\cdot} - X$ | $n_{1\cdot}$ |
| $B=1$ | $n_{\cdot 1} - X$ | $n - n_{\cdot 1} - n_{1\cdot} + X$ | $n_{2\cdot}$ |
| Col. Total | $n_{\cdot 1}$ | $n_{\cdot 2}$ | $n$ |

   (i) Assume that $A$ and $B$ are independent. Find the conditional distribution of $X$ given $n_{1\cdot}$, $n_{\cdot 1}$, and $n$. (6 points)

   (ii) Suppose we observe $n = 5$, $n_{\cdot 1} = 3$, $n_{1\cdot} = 2$, and $X = 0$ and wish to test the hypotheses:

$$H_0 : A \text{ and } B \text{ are independent}$$
$$H_1 : A \text{ and } B \text{ are NOT independent.}$$

   Can we reject $H_0$ under $\alpha = 0.05$? (6 points)

4. Let $X_1, \ldots, X_n \overset{\text{iid}}{\sim} N(\mu, 1)$. Consider the sample mean $\bar{X} = \frac{1}{n}\sum_{i=1}^n X_i$ and sample interquartile range $\text{IQR} = (Q_3 - Q_1)/2$ where $Q_3$ and $Q_1$ are the third and the first quartiles. Show that $\bar{X}$ and IQR are independent. (10 points)

5. Let $X$ be uniformly distributed on $[0, \theta]$, where $\theta \in (0, \infty)$ is an unknown parameter. Given an i.i.d. sample $X_1, \ldots, X_n$, construct two unbiased estimator based on (i) $\sum X_i$ and (ii) $\max_i X_i$. Which one is a better estimator? Justify your answer. (15 points)

6. Let $X$ be a discrete random variable with probability mass function

$$f(x; \alpha, \beta) = \int_0^1 \binom{n}{x} \frac{1}{B(\alpha, \beta)} p^{\alpha+x-1}(1-p)^{\beta+n-x-1} dp, \quad x = 0, 1, \ldots, n$$

where $\alpha, \beta > 0$, $n$ is a known positive integer, and $B(a,b) = \int_0^1 x^{a-1}(1-x)^{b-1}dx$ is the Beta function. Find the expectation and variance of $X$. (15 points)

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7. (i) State the condition under which a binomial distribution can be well approximated by a Poisson distribution. Justify your answer. (8 points)

   (ii) Find that the asymptotic distribution of $Z_\lambda = \frac{X-\lambda}{\sqrt{\lambda}}$ as $\lambda \to \infty$, where $X \sim$ Poisson($\lambda$). (7 points)

8. Suppose $Y_1, \ldots, Y_n$ are independent with $Y_i \sim N(\beta_1 + \beta_2 z_i, \sigma^2)$, where $z_1, \ldots, z_n$ are (fixed not random) covariates not all equal.

   (i) Show that this is an exponential family and specify the natural sufficient statistics for $(\beta_1, \beta_2, \sigma^2)$. (10 points)

   (ii) Calculate the expectation of the sufficient statistics in (i). (5 points)

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