※ 注意：請於試卷內之「非選擇題作答區」標明題號依序作答。

1. (15 points) Determine whether the sequence $(\cos(\pi\sqrt{n^2+n}))_{n=1}^{\infty}$ is convergent or not. Justify your answer.

2. For $x > 1$, define

$$F(x) = \sum_{n=1}^{\infty} n^{-x}.$$

   (a) (5 points) Prove the for any $\delta > 0$ this series converges uniformly on the interval $[1+\delta, \infty)$.

   (b) (5 points) Using (a), show that F is continuous on the interval $(1, \infty)$.

   (c) (5 points) Is F continuously differentiable on $(1, \infty)$? If yes, prove your assertion and compute its derivative (you may leave the answer in terms of a series). If no, explain the reason.

3. (20 points) Let $f : [0,1] \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} \frac{1}{q} & \text{if } x = \frac{p}{q} \in \mathbb{Q} \cap [0,1], \text{ where } \gcd(p,q) = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Prove that f is Riemann integrable and compute $\int_0^1 f(x)\, dx$.

4. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$f(x,y) = \begin{cases} \frac{xy(x^2-y^2)}{x^2+y^2} & \text{if } (x,y) \neq (0,0), \\ 0 & \text{if } x = y = 0. \end{cases}$$

   (a) (12 points) Show that all second order partial derivatives of f exist everywhere.

   (b) (3 points) Is it true that $\dfrac{\partial^2 f}{\partial x \partial y} = \dfrac{\partial^2 f}{\partial y \partial x}$? Justify your answer.

5. Define H to be the following space of sequences:

$$H = \{(x_n)_{n=1}^{\infty} : -2^{-n} \leq x_n \leq 2^{-n} \text{ for all } n \geq 1\}.$$

Consider the function $d : H \times H \to \mathbb{R}$, defined by

$$d((x_n)_{n=1}^{\infty}, (y_n)_{n=1}^{\infty}) = \sup_{n \geq 1} |x_n - y_n|.$$

   (a) (2 points) Show that d is a metric on H.

   (b) (18 points) Prove that every sequence of elements in H has a convergent subsequence with respect to the metric d.

6. (15 points) Suppose that $f : [0,1] \to \mathbb{R}$ is a continuous function such that

$$\int_0^1 (3x+1)^n f(x)\, dx = 0 \quad \text{for all } n \in \mathbb{N} \cup \{0\}.$$

Show that $f(x) = 0$ for all $x \in [0,1]$.

試題隨卷繳回