國立臺北大學113學年度碩士班一般入學考試試題

系（所）組別：統計學系

科　　　　目：數理統計

☐可　☑不可使用計算機

I. Let the joint pdf of random variables X and Y be given by

$$f(x,y) = \begin{cases} cx & 0 < x < y < 1 \\ 0 & \text{elsewhere.} \end{cases}$$

1. (15%) Let Z = X + Y. Derive the CDF of Z.
2. (15%) Compute E(X|Z).
3. (10%) Compute E[E(X|Z)] based on your answer in 2.
4. (5%) Compute E(X) based on the pdf of X.
5. (5%) Suppose you are dealing with an unknown joint pdf f(x, y). Suppose you only have two random samples of size n: one is $X_1, X_2, \dots, X_n$ from the distribution of X, and the other is $W_1, W_2, \dots, W_n$ from the distribution of W, where W = E(X|Z). Can you determine which random sample would allow you to derive a more precise estimator of E(X)? Provide an explanation for your choice.

II.

1. (20%) Let $X_1, X_2, \dots, X_n$ be a random sample from Bernoulli(p). Consider two estimators, $\hat{p}_U = \bar{X}$ and $\hat{p}_B = \frac{n}{n+\sqrt{n}}\bar{X} + \frac{\sqrt{n}}{n+\sqrt{n}}\frac{1}{2}$.
   a. (5%) Find the limiting distribution of $\sqrt{n}(\hat{p}_U - p)$ and $\sqrt{n}(\hat{p}_B - p)$, respectively.
   b. (5%) Show that $\hat{p}_U$ and $\hat{p}_B$ are the consistent estimators of p.
   c. (5%) Find a consistent estimator of $\frac{1}{p}$.
   d. (5%) Use $n = 1$ to show that no unbiased estimator of $\frac{1}{p}$ exists.

2. (10%) Let $X_1, X_2, \dots, X_n$ be a random sample from $U(0, \theta)$.
   a. (5%) Find the shortest length $100(1-\alpha)\%$ confidence interval for θ in the class $C(\boldsymbol{X}) := \left[\frac{X_{(n)}}{a}, \frac{X_{(n)}}{b}\right]$.
   b. (5%) Prove that $P_\theta[\theta' \in C(\boldsymbol{X})] < 1 - \alpha$ for $\theta' \neq \theta$.

3. (15%) Let $X_1, X_2, \dots, X_n$ be a random sample from the pdf

$$f(x|\theta) = \theta x^{-2},\ 0 < \theta \le x \le \infty.$$

   a. (5%) Find a sufficient statistic of θ.
   b. (5%) Find the maximum likelihood estimator (MLE) of θ.
   c. (5%) Find the method of moments estimator of θ.

4. (5%) Consider the testing problem $H_0: \theta = \theta_0$ versus $H_1: \theta \neq \theta_0$ based on *i.i.d.* $X_1, X_2, \dots, X_n$ from $U(0, \theta)$. Show that the uniformly most powerful (UMP) test with rejection region $X_{(n)} > \theta_0$ or $X_{(n)} \le \theta_0 \alpha^{1/n}$ is a likelihood ratio (LR) test with size α.

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