國立中山大學113學年度
碩士班暨碩士在職專班招生考試試題

科目名稱：線性代數【通訊所碩士班甲組】

—作答注意事項—

考試時間：100分鐘

- **考試開始鈴響前不得翻閱試題，並不得書寫、劃記、作答。**請先檢查答案卷（卡）之應考證號碼、桌角號碼、應試科目是否正確，如有不同立即請監試人員處理。
- **答案卷限用藍、黑色筆(含鉛筆)書寫、繪圖或標示**，可攜帶橡皮擦、無色透明無文字墊板、尺規、修正液（帶）、手錶(未附計算器者)。每人每節限使用一份答案卷，請衡酌作答。
- 答案卡請以2B鉛筆劃記，不可使用修正液（帶）塗改，未使用2B鉛筆、劃記太輕或污損致光學閱讀機無法辨識答案者，後果由考生自負。
- 答案卷（卡）應保持清潔完整，不得折疊、破壞或塗改應考證號碼及條碼，亦不得書寫考生姓名、應考證號碼或與答案無關之任何文字或符號。
- 可否使用計算機請依試題資訊內標註為準，如「可以」使用，廠牌、功能不拘，唯不得攜帶書籍、紙張（應考證不得做計算紙書寫）、具有通訊、記憶、傳輸或收發等功能之相關電子產品或其他有礙試場安寧、考試公平之各類器材入場。
- 試題及答案卷（卡）請務必繳回，未繳回者該科成績以零分計算。
- 試題採雙面列印，考生應注意試題頁數確實作答。
- 違規者依本校招生考試試場規則及違規處理辦法處理。

一、單選題 （每題5分）

1. (5%) Which of the following statement is **False**?
(A) If $\mathbf{B}$ is obtained from a matrix $\mathbf{A}$ by several elementary row operations, then $\text{rank}(\mathbf{B}) = \text{rank}(\mathbf{A})$.
(B) Row operations on a matrix $\mathbf{A}$ can change the linear dependence relations among the rows of $\mathbf{A}$.
(C) A change-of-coordinates matrix is always invertible.
(D) If $\mathbf{A}$ is $m \times n$ and $\text{rank}\mathbf{A} = m$, then the linear transform $\mathbf{x} \mapsto \mathbf{Ax}$ is one-to-one.
(E) If $\mathbf{A}$ is $m \times n$ and linear transformation $\mathbf{x} \mapsto \mathbf{Ax}$ is onto, then $\text{rank}\mathbf{A} = m$.

2. (5%) Which of the following statement is **False**?
(A) If an augmented matrix $[\mathbf{A}\ \mathbf{b}]$ is transformed into $[\mathbf{C}\ \mathbf{d}]$ by elementary row operations, then the equations $\mathbf{Ax} = \mathbf{b}$ and $\mathbf{Cx} = \mathbf{d}$ have exactly the same solution sets.
(B) If a system $\mathbf{Ax} = \mathbf{b}$ has more than one solution, then so does the system $\mathbf{Ax} = \mathbf{0}$.
(C) If matrices $\mathbf{A}$ and $\mathbf{B}$ are row equivalent, they have the same reduced echelon form.
(D) If $\mathbf{A}$ is an $m \times n$ matrix and the equation $\mathbf{Ax} = \mathbf{b}$ is consist for every $\mathbf{b}$ in $\mathbb{R}^m$, then $\mathbf{A}$ has m pivot column.
(E) If $\mathbf{A}$ is an $m \times n$ matrix and the equation $\mathbf{Ax} = \mathbf{b}$ is consistent for some $\mathbf{b}$, then the columns of $\mathbf{A}$ span $\mathbb{R}^m$.

3. (5%) Which of the following statement is **False**?
(A) If $\mathbf{A}$ and $\mathbf{B}$ are row equivalent $m \times n$ matrices and if the columns of $\mathbf{A}$ span $\mathbb{R}^m$, then so do the columns of $\mathbf{B}$.
(B) In some cases, it is possible for four vectors to span $\mathbb{R}^5$.
(C) If $\mathbf{u}$ and $\mathbf{v}$ are in $\mathbb{R}^m$, then $-\mathbf{u}$ is in $\text{Span}\{\mathbf{u}, \mathbf{v}\}$.
(D) If $\mathbf{A}$ is a 6×5 matrix, the linear transformation $\mathbf{x} \mapsto \mathbf{Ax}$ cannot map $\mathbb{R}^5$ onto $\mathbb{R}^6$.
(E) A linear transform is a function.

4. (5%) Which of the following statement is **False**?
(A) If $\mathbf{A}$ and $\mathbf{B}$ are $m \times n$, then both $\mathbf{AB}^T$ and $\mathbf{A}^T\mathbf{B}$ are defined.
(B) Left-multiplying a matrix $\mathbf{B}$ by a diagonal matrix $\mathbf{A}$, with nonzero entries on the diagonal, scales the rows of $\mathbf{B}$.
(C) If $\mathbf{BC} = \mathbf{BD}$, then $\mathbf{C} = \mathbf{D}$.
(D) If $\mathbf{AB} = \mathbf{BA}$ and if $\mathbf{A}$ is invertible, then $\mathbf{A}^{-1}\mathbf{B} = \mathbf{BA}^{-1}$.
(E) An elementary $n \times n$ matrix has either n or $n+1$ nonzero entries.

5. (5%) Which of the following statement is **False**?
(A) If $\mathbf{B}$ is formed by adding to one row of $\mathbf{A}$ a linear combination of other rows, then $\det(\mathbf{A}) = \det(\mathbf{B})$.
(B) $\det(\mathbf{A}^T\mathbf{A}) \geq 0$.
(C) If $\mathbf{A}^3 = \mathbf{0}$, then $\det(\mathbf{A}) = 0$.
(D) $\det(-\mathbf{A}) = -\det(\mathbf{A})$.
(E) If $\mathbf{A}$ is invertible, then $\det(\mathbf{A})\det(\mathbf{A}^{-1}) = 1$.

6. (5%) The dimension of the subspace

$$H = \left\{ \begin{bmatrix} a-3b+6c \\ 5a+4d \\ b-2c-d \\ a-2b+4c-d \end{bmatrix} : a,b,c,d \in \mathbb{R} \right\}$$

is
(A) 1.
(B) 2.
(C) 3.
(D) 4.
(E) 5.

7. (5%) Let
$\mathbf{A} = \begin{bmatrix} 0.4 & -0.3 \\ 0.4 & 1.2 \end{bmatrix}$. As $k \to \infty$, we obtain $\mathbf{A}^k$
(A) $\begin{bmatrix} -0.5 & -1.75 \\ 1.0 & 1.50 \end{bmatrix}$.
(B) $\begin{bmatrix} -0.75 & -0.5 \\ 1.0 & 1.50 \end{bmatrix}$.
(C) $\begin{bmatrix} -0.5 & 1.50 \\ 1.0 & -0.75 \end{bmatrix}$.
(D) $\begin{bmatrix} -1.5 & -0.75 \\ 1.0 & 2.50 \end{bmatrix}$.
(E) $\begin{bmatrix} -0.5 & -0.75 \\ 1.0 & 1.50 \end{bmatrix}$.

8. (5%) Let $\mathbf{J}$ be the $n \times n$ matrix of all 1's, and consider $\mathbf{A} = (a-b)\mathbf{I} + b\mathbf{J}$; that is

$$\mathbf{A} = \begin{bmatrix} a & b & b & \dots & b \\ b & a & b & \dots & b \\ b & b & a & \dots & b \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ b & b & b & \dots & a \end{bmatrix}.$$

Then the eigenvalues of $\mathbf{A}$ are
(A) $a+b$, and $a+(n-1)b$.
(B) $a-nb$, and $a+nb$.
(C) $a-b$, and $a+(n-1)b$.
(D) $a-2b$, and $a+nb$.
(E) $a+b$, and $a-(n-1)b$.

9. (5%) Let $\mathbf{A}$ and $\mathbf{B}$ be 4×4 matrices, with $\det \mathbf{A} = -1$ and $\det \mathbf{B} = 4$. Then,
$\det \mathbf{B}^{-1}\mathbf{A}\mathbf{B} + \det \mathbf{A}^T\mathbf{A} + \det 2\mathbf{A} =$
(A) -12.
(B) -14.
(C) -16.
(D) -18.
(E) -20.

10. (5%) The determinant of

$$\mathbf{A}=\begin{bmatrix} 3a & -7 & 8 & 9 & -6 \\ 0 & 2 & -5 & 7 & 7 \\ 0 & 0 & 1 & 5 & 0 \\ 0 & 0 & 2 & 4 & -1 \\ 0 & 0 & 0 & 2 & 0 \end{bmatrix}$$

is

(A) 0.
(B) $12a$.
(C) $13a$.
(D) $14a$.
(E) $15a$.

二 、問答計算題（請於答案卷作答）

1. (15%) Consider the following matrix

$$\mathbf{A}=\begin{bmatrix} -1 & 0 \\ 1 & -1 \\ 1 & 1 \end{bmatrix}$$

(a) (5%) Please find the eigenvalues and eigenvectors of the matrix $\mathbf{A}\mathbf{A}^T$, where $\mathbf{A}^T$ is the transport of $\mathbf{A}$.

(b) (10%) Please calculate the singular value decomposition (SVD) of $\mathbf{A}$.

2. (15%) Let $\mathbf{U}$ and $\mathbf{V}$ be two $m \times m$ positive definite matrices.

(a) (10%) Find a $m \times 1$ complex vector $\mathbf{b}$, such that

$$Q=\frac{\mathbf{b}\mathbf{U}\mathbf{b}^H}{\mathbf{b}\mathbf{V}\mathbf{b}^H}$$

is maximized

(b) (5%) What is the maximum value of Q in (a)?

3. (10%) Show that if the set $\{\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}\}$ is linearly independent, then so is the set $\{\boldsymbol{u}, \boldsymbol{u}+\boldsymbol{v}, \boldsymbol{u}+\boldsymbol{v}+\boldsymbol{w}\}$.

4. (10%) If the columns of a $m \times n$ matrix $\mathbf{A}$ are linearly independent, show that the projection of a $m \times 1$ vector $\mathbf{A}$ on to the column space of $\mathbf{A}$ is

$$\mathbf{p}=\mathbf{A}(\mathbf{A}^T\mathbf{A})^{-1}\mathbf{A}^T\mathbf{b}$$