國立中山大學113學年度
碩士班暨碩士在職專班招生考試試題

科目名稱：機率與統計【應數系碩士班甲組】

—作答注意事項—

考試時間：100分鐘

- **考試開始鈴響前不得翻閱試題，並不得書寫、劃記、作答。**請先檢查答案卷（卡）之應考證號碼、桌角號碼、應試科目是否正確，如有不同立即請監試人員處理。
- **答案卷限用藍、黑色筆(含鉛筆)書寫、繪圖或標示**，可攜帶橡皮擦、無色透明無文字墊板、尺規、修正液（帶）、手錶(未附計算器者)。每人每節限使用一份答案卷，請衡酌作答。
- 答案卡請以2B鉛筆劃記，不可使用修正液（帶）塗改，未使用2B鉛筆、劃記太輕或污損致光學閱讀機無法辨識答案者，後果由考生自負。
- 答案卷（卡）應保持清潔完整，不得折疊、破壞或塗改應考證號碼及條碼，亦不得書寫考生姓名、應考證號碼或與答案無關之任何文字或符號。
- 可否使用計算機請依試題資訊內標註為準，如「可以」使用，廠牌、功能不拘，唯不得攜帶書籍、紙張（應考證不得做計算紙書寫）、具有通訊、記憶、傳輸或收發等功能之相關電子產品或其他有礙試場安寧、考試公平之各類器材入場。
- 試題及答案卷（卡）請務必繳回，未繳回者該科成績以零分計算。
- 試題採雙面列印，考生應注意試題頁數確實作答。
- 違規者依本校招生考試試場規則及違規處理辦法處理。

國立中山大學 113 學年度碩士班暨碩士在職專班招生考試試題

科目名稱：機率與統計【應數系碩士班甲組】　　　　題號：424006
※本科目依簡章規定「不可以」使用計算機(問答申論題)　　　　共1頁第1頁

- Please **answer the questions in order** and **write down the question number** for **each** question. If you are not able to answer the question, leave it blank.
- Notations: i.i.d., independent and identically distributed; cdf, cumulative distribution function; pdf, probability density function; Z is a standard normal random variable.

1. (15%) Let X and Y be two nondegenerate random variables with finite variances. It is readily seen that $\hat{\theta} = X$ is an unbiased estimator for $\theta = E(X)$. Instead of using $\hat{\theta}$, we consider an alternative estimator $\hat{\theta}_{\alpha,\beta} = X - \beta(Y - \alpha)$, where $-\infty < \alpha < \infty$ and $\beta \neq 0$ are parameters. In the subsequent questions, please express your answers in terms of E, Var, and Cov.
   (a) (5%) Find α such that $\hat{\theta}_{\alpha,\beta}$ is an unbiased estimator.
   (b) (5%) Find β such that $\mathrm{Var}(\hat{\theta}_{\alpha,\beta})$ is minimized.
   (c) (5%) Find the percentage of variance reduction obtained by using $\hat{\theta}_{\alpha,\beta}$ instead of $\hat{\theta}$, that is, $\{\mathrm{Var}(\hat{\theta}) - \mathrm{Var}(\hat{\theta}_{\alpha,\beta})\}/\mathrm{Var}(\hat{\theta}) \times 100\%$ based on α and β you found in (a) and (b).

2. (30%) Let $X_1, \ldots, X_n$ be i.i.d. uniform random variables on the unit interval $[0,1]$.
   (a) (5%) Find the joint cdf of the random vector (X_1, X_1).
   (b) (5%) Find the joint cdf of the random vector $(X_1, 1 - X_1)$.
   (c) (10%) Find the pdf of $Y = \prod_{i=1}^{n} X_i$.
   (d) (10%) Find the pdf of $Y = \max(X_1, \ldots, X_n) - \min(X_1, \ldots, X_n)$.

3. (15%) Let $X_1, \ldots, X_n$ be i.i.d. continuous random samples from the cdf $F(x)$. Define the empirical cdf $F_n(x) = n^{-1} \sum_{i=1}^{n} I(X_i \leq x)$, where $I(.)$ is the indicator function. Let s be a constant.
   (a) (5%) Find the asymptotic distribution of $n^{1/2}\{F_n(s) - F(s)\}$.
   (b) (5%) Find an asymptotic 95% confidence interval for $F(s)$ based on the result in (a). Please express your answer by using z_α that satisfies $P(Z > z_\alpha) = \alpha$.
   (c) (5%) Find the covariance $\mathrm{Cov}\{F_n(s), F_n(t)\}$, where t is another constant and $s \neq t$.

4. (10%) Let $X_1, \ldots, X_n$ be i.i.d. random samples from the inverse Gaussian pdf
$$f_{\mu,\lambda}(x) = \left(\frac{\lambda}{2\pi x^3}\right)^{1/2} \exp\left\{-\frac{\lambda(x-\mu)^2}{2\mu^2 x}\right\}, \quad x > 0, \mu > 0, \lambda > 0.$$
   Find the maximum likelihood estimators for μ and λ.

5. (10%) Let $X_1, \ldots, X_n$ be i.i.d. normal random variables with mean μ_X and variance σ^2. Also, let $Y_1, \ldots, Y_m$ be i.i.d. normal random variables with mean μ_Y and variance σ^2. Suppose that X_i's and Y_i's are independent and the parameters μ_X, μ_Y, and σ^2 are unknown. Find the two-tailed t-test for testing the null hypothesis $H_0: \mu_X = \mu_Y$ against the alternative hypothesis $H_1: \mu_X \neq \mu_Y$ with the significance level α. Please express your answer by using $t_{\alpha,\nu}$ that satisfies $P(T_\nu > t_{\alpha,\nu}) = \alpha$, where T_ν is a Student t random variable with ν degrees of freedom.

6. (10%) Suppose that there are two people. Each of them tosses a fair coin n times. Find the probability that they will toss the same number of heads. You must simplify your answer.

7. (10%) Prove that $P(Z > z) < \frac{1}{z}\phi(z)$ if $z > 0$, where $\phi(z)$ is the pdf of Z.

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