多重選擇題，共20題，每題5分， 每題每一選項(ABCDE)單獨計分，每一選項個別分數為1分
答錯一個選項倒扣1分，倒扣至本大題(即多重選擇題)0分為止

1. Let $M(t) = E[e^{tX}]$ be the moment generating function of a random variable X, which is defined in a neighborhood of 0. Define $\Psi(t) = \ln M(t)$. Let $M^{(n)}(t)$ and $\Psi^{(n)}(t)$ be the nth derivatives of $M(t)$ and $\Psi(t)$ respectively. Which of the following statements is/are true?

(A) The nth moment $E[X^n]$ of X is $M^{(n)}(t)\big|_{t=0}$.

(B) The nth moment $E[X^n]$ of X is $\frac{1}{n!}M^{(n)}(t)\big|_{t=0}$.

(C) The variance of X is $\Psi^{(2)}(t)\big|_{t=0}$.

(D) The variance of X is $\frac{1}{2}\Psi^{(2)}(t)\big|_{t=0}$.

(E) None of the above.

2. Consider two random variables X and Y with a joint probability desity function

$$f(x,y) = \begin{cases} \lambda^2 e^{-\lambda(x+y)}, & \text{if } x \geq 0,\ y \geq 0 \\ 0, & \text{otherwise,} \end{cases}$$

where $\lambda > 0$. Which of the following statements is/are true?

(A) The moment generating function $M_X(t) = E[e^{tX}]$ of X is $\frac{\lambda^2}{\lambda^2 - t}$.

(B) The moment generating function $M_Y(t)$ of Y is $\frac{\lambda}{\lambda - t}$.

(C) The joint moment generating function $M_{X,Y}(s,t) = E[e^{sX+tY}]$ of X and Y is $\frac{\lambda^2}{(\lambda-s)(\lambda-t)}$.

(D) The joint moment generating function $M_{X,Y}(s,t)$ of X and Y is $\frac{\lambda}{\lambda-(s+t)}$.

(E) The joint moment $E[X^n Y^m]$ of X and Y is $(n!m!)/\lambda^{n+m}$.

3. Consider a sequence $\{X_n\}_{n=1}^{\infty}$ of statistically independent and identically distributed random variables with common mean μ and variance σ^2. Let

$$\overline{X}_n \triangleq \frac{1}{n}\sum_{k=1}^{n} X_k$$

be the sample mean for each $n = 1, 2, \ldots$. Which of the following statements is/are true?

(A) The expectation $E[\overline{X}_n]$ of the sample mean $\overline{X}_n$ is μ/n.

(B) The variance $\mathrm{Var}(\overline{X}_n)$ of $\overline{X}_n$ is σ^2/n^2.

(C) Fix an $\epsilon > 0$. For any $\delta > 0$, the probability of the event $(\mu - \epsilon < \overline{X}_n < \mu + \epsilon)$ can be lower bounded by $1 - \delta$ with a sufficiently large n.

(D) The event $(\lim_{n\to\infty} \overline{X}_n = 0)$ occurs with probability one.

(E) The event $(\lim_{n\to\infty} \overline{X}_n = \mu)$ occurs with probability one.

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4. Let Z be a standard normal distribution with a cumulative distribution function $\Phi(z) = P(Z \le z)$. Let α be a number in the interval $(0,1)$. Define z_α to be the real number such that $P(Z > z_\alpha) = \alpha$. Also let $\{X_n\}_{n=1}^{\infty}$ be a sequence of statistically independent and identically distributed random variables with common mean μ and variance σ^2. Which of the following statements is/are true?

(A) For any $\alpha \in (0,1)$, we have $P(|Z| > z_{\alpha/2}) = \alpha$.

(B) For any $\alpha \in (0,1)$, we have $P(|Z| > z_{\alpha/2}) = \alpha/2$.

(C) For a sufficiently large n, a good solution for the unknow x satisfying

$$P(X_1 + X_2 + \cdots + X_n > x) = \alpha$$

with $\alpha \in (0,1)$ is $z_\alpha(n\mu + \sigma\sqrt{n})$.

(D) For a sufficiently large n, a good solution for the unknow x satisfying

$$P(X_1 + X_2 + \cdots + X_n > x) = \alpha$$

with $\alpha \in (0,1)$ is $n\mu + z_\alpha \sigma\sqrt{n}$.

(E) For a sufficiently large n, a good approximation to the probability $P(X_1 + X_2 + \cdots + X_n \le a)$ for a real number a is $\Phi\left(\frac{a-n\mu}{\sigma\sqrt{n}}\right)$.

5. The joint probability density function of X and Y is given by

$$f(x,y) = \begin{cases} ce^{-x}, & if\ x \ge 0, |y| < x \\ 0, & otherwise \end{cases}$$

where c is a constant. Which of the following statements is/are true?

(A) $c = \frac{1}{2}$.

(B) $f_{Y|X}(y|x) = 2x, -x < y < x$.

(C) $E(Y|X = x) = 0$.

(D) $Var(Y|X = x) = 3x^2$.

(E) $E(Y^2|X = x) = \frac{x^2}{3}$.

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6. Let X, Y and Z be continuous random variables with the following joint probability density function:

$$f(x,y,z) = \begin{cases} x^2 e^{-x(1+y+z)}, & \text{if } x,y,z > 0 \\ 0 & , \text{otherwise} \end{cases}$$

Which of the following statements is/are true?

(A) $f_Z(z) = \frac{1}{(1+z)^2}$, $z > 0$.

(B) $f_Y(y) = \frac{1}{(1+y)^2}$, $y > 0$.

(C) X and Y are independent.

(D) X, Y and Z are pairwise independent.

(E) X, Y and Z independent.

7. Suppose that a group of 20 students have received 4 tickets of a special event. Suppose that 4 students will be selected by random to win the tickets. Suppose that among the 20 students, 15 are male and 5 are female. Here it is assumed that the random selection is fair. What is the probability that among the 4 selected students, 3 are male and 1 is female? (Choose the closest answer.)

(A) 22%

(B) 47%

(C) 63%

(D) 84%

(E) 92%

8. Suppose that Ω is a sample space and A and B are subsets of Ω. Suppose that P is a probability function defined on F, which is a σ-field on Ω. Which following equations are correct?

(A) $P(\emptyset)=0$

(B) $P(A) + P(A^c) = 1$

(C) $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

(D) For $A \subset B$, $P(A) \le P(B)$

(E) $0 \le P(A) \le 1$

9. Let X and Y be two continuous random variables. Which of the following is TRUE.

(A) $E[XY] = E[X]E[Y]$

(B) $f_{X+Y}(x+y) = f_X(x) + f_Y(Y)$

(C) $var(X+Y) = var(X) + var(Y) - 2E[XY]$

(D) $E[X^2 + Y^2] = E[X^2] + E[Y^2]$

(E) None of the above

10. Suppose X is uniformly distributed over $[0,4]$ and Y is uniformly distributed over $[0,1]$. Assume X and Y are independent. $P(max(X,Y) > 2)$ is equal to?

(A) 0
(B) 1/4
(C) 1/2
(D) 3/4
(E) 1

11. Let $A = \begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix}$, which of the following descriptions are true?

(A) A is invertible.
(B) A is diagonalizable with real-valued eigenvalues.
(C) A is normal.
(D) A is Hermitian.
(E) A is unitary.

12. Which of the following properties of the "determinant" of an n x n matrix are incorrect?

(A) We can use "cofactor expansion" to calculate the determinant along any row or column.
(B) For an invertible matrix, its determinant cannot be 0.
(C) For two n x n matrices: A and B, $\det(AB) = \det(A)\cdot\det(B)$.
(D) If we apply elementary row operations to calculate a matrix's determinant, it requires less multiplications than using cofactor expansion.
(E) For an upper triangular matrix, its determinant is the product of the diagonal elements.

13. Which of the following properties on subspace are correct?

(A) Every subspace contains infinite number of vectors.
(B) $\{0\}$ is a subspace of any vector space.
(C) The basis of a subspace can be extended to a basis of the vector space that contains this subspace.
(D) The dimension of a subspace is less than the vector space that contains this subspace.
(E) The interception of any two subspaces contains at least one vector.

14. For a linear transformation **T**: $R^3 \to R^3$ defined by: $\mathbf{T}(x, y, z) = (3x + 2y, -2x + 3y, 5z)$, which of the following statements are correct?

(A) The basis of the kernel (null space) of **T** is $\{0\}$.
(B) **T** is one to one.
(C) **T** is onto.
(D) **T** is invertible.
(E) **T** is diagonalizable.

15. For a 5x5 matrix: $A(t)=\begin{pmatrix} 6 & 1 & -2 & 0 & 0 \\ 2 & 2 & 3 & 2 & 2 \\ 4 & 3 & 1 & 3 & 4 \\ -1 & 3 & 2 & 0 & 0 \\ 2 & 1 & -1+\cos t & 1 & 2 \end{pmatrix}$, which value of t will make $\det(A)=0$?

(A) $-\pi$
(B) $-\pi/2$
(C) 0
(D) $\pi/2$
(E) π

16. What are the corresponding eigen vectors for the matrix A having the following form.

$$A=\begin{pmatrix} 1 & -2 & 8 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix},$$

(A) $\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}1\\1\\0\end{pmatrix}, \begin{pmatrix}4\\0\\1\end{pmatrix}$

(B) $\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}1\\-1\\0\end{pmatrix}, \begin{pmatrix}4\\0\\1\end{pmatrix}$

(C) $\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}1\\1\\0\end{pmatrix}, \begin{pmatrix}-4\\0\\1\end{pmatrix}$

(D) $\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}1\\-1\\0\end{pmatrix}, \begin{pmatrix}-4\\0\\1\end{pmatrix}$

(E) $\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}, \begin{pmatrix}4\\0\\1\end{pmatrix}$

17. Following the previous question, evaluate A^{2301}.

(A) $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$,

(B) $\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$,

(C) $\begin{pmatrix} 1 & 2 & 8 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$,

(D) $\begin{pmatrix} 1 & -2 & 8 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$,

(E) $\begin{pmatrix} 1 & -2 & 0 \\ 0 & -1 & 8 \\ 0 & 0 & -1 \end{pmatrix}$,

18. For a rectangle matrix $M = \begin{pmatrix} 4 & 11 & 14 \\ 8 & 7 & -2 \end{pmatrix}$, how to find the corresponding singular value?

(A) By finding the eigen-values of $M = \begin{pmatrix} 4 & 11 & 14 \\ 8 & 7 & -2 \\ 0 & 0 & 0 \end{pmatrix}$,

(B) By finding the eigen-values of $MM = M^2$

(C) By finding the eigen-values of $\lim_{n\to\infty} M^n$

(D) By finding the eigen-values of $\frac{M}{|M|}$

(E) By finding the eigen-values of MM^t

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19. Following the previous question, find the largest singular value of the matrix M

(A) 600

(B) 360

(C) $10\sqrt{6}$

(D) $6\sqrt{10}$

(E) $3\sqrt{6}$

20. Which of the following number β can make the matrix $\boldsymbol{B}$ being positive definite,

$$\boldsymbol{B} = \begin{pmatrix} \beta & 1 & 1 \\ 1 & \beta & 1 \\ 1 & 1 & \beta \end{pmatrix}$$

(A) -1

(B) 0

(C) 1

(D) 2

(E) i