# 國立成功大學

# 113學年度碩士班招生考試試題

編　號：35

系　所：數學系應用數學

科　目：線性代數

日　期：0202

節　次：第 1 節

備　註：不可使用計算機

※ 考生請注意：本試題不可使用計算機。　請於答案卷(卡)作答，於本試題紙上作答者，不予計分。

All vector spaces are finite dimensional over the base field.
Let $K$ be a field. The space of $n \times n$ matrices with entries in $K$ is denoted by $\mathcal{M}_n(K)$.
An operator on a $K$-vector space $V$ is a $K$-linear transformation from $V$ to itself.

1. (10 points) Let $M \in \mathcal{M}_{2n}(\mathbb{R})$ be a matrix that satisfies $M^T \Omega M = \Omega$ where
$$\Omega = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}.$$
Show that $M$ is invertible with inverse $M^{-1} = \Omega^T M^T \Omega$.

2. Let $A, B, C, D \in \mathcal{M}_n(K)$ and let $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathcal{M}_{2n}(K)$.
   (a) (10 points) When $C = 0$, show that
   $$M = \begin{pmatrix} I_n & 0 \\ 0 & D \end{pmatrix} \begin{pmatrix} I_n & B \\ 0 & I_n \end{pmatrix} \begin{pmatrix} A & 0 \\ 0 & I_n \end{pmatrix}.$$
   Deduce that $\det M = \det A \cdot \det D$.
   (b) (15 points) Show that if $A$ is invertible and $AC = CA$, then $\det M = \det(AD - CB)$.

3. (15 points) Let
$$M = \begin{pmatrix} 5 & 4 & 2 & 1 \\ 0 & 1 & -1 & -1 \\ -1 & -1 & 3 & 0 \\ 1 & 1 & -1 & 2 \end{pmatrix}.$$
Find $D, U \in \mathcal{M}_4(\mathbb{R})$ such that $M = DU = UD$, $D$ is diagonalizable, and $U - I_4$ is nilpotent.

4. (10 points) Let $A, B, C$ be in $\mathcal{M}_n(K)$. Show that the traces
$$\operatorname{Tr} A[B, C] = \operatorname{Tr} B[C, A] = \operatorname{Tr} C[A, B]$$
where $[M, N] = MN - NM$ for any $M, N \in \mathcal{M}_n(K)$.

5. (15 points) Let $V$ be an inner product space over $\mathbb{C}$. An operator $T$ on $V$ is *positive* if it is self-adjoint and $\langle Tv, v \rangle \geq 0$ for all $v \in V$. Show that an operator $T$ on $V$ is positive if and only if there exists a unique positive operator $R$ on $V$ such that $R^2 = T$.

6. Let $V$ be a $\mathbb{C}$-vector space of dimension $n$ and let $T, U$ be operators on $V$.
   Assume that the minimal polynomial of $T$ has degree $n$ and that $TU = UT$.
   Let $\mathbb{C}[x, y]$ be the polynomial ring of two variables. Prove the following statements.
   (a) (10 points) There exists $v \in V$ such that $\{v, T(v), \ldots, T^{n-1}(v)\}$ form a basis for $V$.
   (b) (10 points) The two vector spaces
   $$\mathbb{C}[T, U] := \{p(T, U) \mid p(x, y) \in \mathbb{C}[x, y]\} \text{ and } \mathbb{C}[T] := \{p(T) \mid p(x) \in \mathbb{C}[x]\}$$
   are equal.
   (c) (5 points) The dimension of $\mathbb{C}[T, U]$ is less than or equal to $n$.