國立成功大學

113學年度碩士班招生考試試題

編　號： 225

系　所： 統計學系

科　目： 數學

日　期： 0202

節　次： 第 1 節

備　註： 不可使用計算機

※ 考生請注意：本試題不可使用計算機。　請於答案卷(卡)作答，於本試題紙上作答者，不予計分。

I. 簡答題（**請依題號於第一頁依序寫下答案，不用計算過程**）

1. (10%) The probability density function of a normal distribution with mean μ and standard deviation σ is given by the formula

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}.$$

Suppose a sample has a true weight of 100 grams, and a measurement error, following a normal distribution with a mean of 0 and a standard deviation of $1/\sqrt{2}$ grams, is introduced. Given that $\sqrt{\pi} = 1.772$, find the approximate probability that the measured weight falls between 100 and 101 grams correct to two decimal places.

2. (10%) Determine the volume of the solid bounded by the surface defined by the equation

$$(x^2 + y^2 + z^2)^2 - 2z(x^2 + y^2) = 0.$$

3. (10%, 5% for each) Evaluate the following integrals

(a) $\displaystyle\int_0^{\pi/4} \sec^3\theta \, d\theta$

(b) $\displaystyle\int_0^1 \int_y^1 (1 + x^2 + y^2)^{-\frac{3}{2}} \, dxdy$

4. (10%) Find the specific function $f(x)$ satisfying the following condition:

$$f(x) = 1 + \int_0^x \frac{t\sin(t)}{f^2(t)} dt.$$

5. (10%, 5% for each) Evaluate

(a) $\displaystyle\lim_{n\to\infty}\left(\frac{1}{\sqrt{n^2+1^2}} + \frac{1}{\sqrt{n^2+2^2}} + \cdots + \frac{1}{\sqrt{n^2+n^2}}\right)$

(b) $\displaystyle\lim_{n\to\infty}\left(\frac{1}{n^2+1} + \frac{2}{n^2+2} + \cdots + \frac{n}{n^2+n}\right)$

6. (10%, 2% for each) True or False (n and m are positive integers)

(a) Let $A \in M_{n\times n}(\mathbb{R})$. If $\sqrt{2}$ is an eigenvalue value of A, then $\lim_{k\to\infty} A^k$ does not exist.

(b) Let T be the transformation $T: P(\mathbb{R}) \to P(\mathbb{R})$ given by

$$T(f(x)) = \int_0^x f(t)dt.$$

Then, T is linear, one-to-one, and onto. ($P(\mathbb{R})$ is the set of all polynomials with coefficients from $\mathbb{R}$.)

(c) Let $A \in M_{m\times n}(\mathbb{R})$. If the rank of A is zero, then A is the zero matrix.

(d) Every orthonormal set is linearly independent.

(e) Let $A \in M_{n\times n}(\mathbb{R})$. If $A^3 = A$, then A is diagonalizable.

II. 證明題（請依題號從第二頁開始寫下答案，詳列過程）

1. Let A and B be matrices in $M_{n\times n}(\mathbb{R})$, where n is a positive integer.

(a) (10%) Prove that if A and B are similar, then A and B have the same eigenvalues.

(b) (15%) Prove that if A and B are symmetric positive definite, then

$$\det(A)\det(B) \le \left(\frac{\operatorname{trace}(AB)}{n}\right)^n.$$

2. (15%) Let T be a linear transformation $T: V_1 \to V_2$ where V_1 and V_2 are vector spaces. Prove that T is one-to-one if and only if the null space of T is $\{0\}$.