國立政治大學 113 學年度 碩士班暨碩士在職專班 招生考試試題

考 試 科 目	基礎數學	系 所 別	統計學系	考試時間	2 月 6 日(二) 第 一 節

Part I: Calculus

1. **(20%)** Theorem 1. A real valued function $f(x)$ is differentiable at a point $x = a$ implies that $f(x)$ is continuous at $x = a$ (i.e., $\lim_{x \to a} f(x) = f(a)$).
   (a) (5%) State the formal ϵ-δ definition of a limit.
   (b) (5%) State the definition for $f(x)$ being differentiable at $x = a$.
   (c) (10%) Prove Theorem 1 from using the aforementioned ϵ-δ definition of a limit.
2. **(10%)** Find the following integrals:
   (a) (5%) $\int \arctan(x)dx$
   (b) (5%) $\int \sin^2(x)dx$
3. **(10%)** Find the following limits:
   (a) (5%) $\lim_{n \to \infty} \left(n\sqrt[n]{x} - n\right)$
   (b) (5%) $\lim_{(x,y) \to (0,0)} \frac{x^2 y}{x^2 + y^2}$
4. **(10%)** Consider a positive series a_n satisfying $\lim_{n \to \infty} a_n = \infty$. Suppose that $a_n(x_n - 1) \to c$ as $n \to \infty$, where x_n is a sequence of real values and c is a real number.
   (a) (5%) State the Taylor series of an infinitely differentiable real-valued function $f(x)$ at $x = a$.
   (b) (5%) Find the limit of $a_n^2(\log(x_n) - x_n + 1)$ as $a_n \to \infty$. Here $\log(e) = 1$.

Part II: Linear Algebra

(Please indicate with a tilde (~) below all the variables that are vectors or matrices on your answer sheet)

5. **(10%)** Consider $\boldsymbol{A}$ an $n \times n$ square matrix.
   (a) (5%) State the definition of the left inverse and right inverse matrices of the matrix $\boldsymbol{A}$.
   (b) (5%) Prove that if $\boldsymbol{A}$ has a left inverse $\boldsymbol{B}$ and a right inverse $\boldsymbol{C}$, then $\boldsymbol{B} = \boldsymbol{C}$.
6. **(10%)** Let V be a vector space and S be a subset of V contains only finite many vectors $\boldsymbol{\alpha}_1, \ldots, \boldsymbol{\alpha}_n$.
   (a) (5%) State the condition(s) for S being linearly dependent.
   (b) For each of the following statements, determine whether it is TRUE (O) or FALSE (X). (Do not give explanation):
      i. (1%) Any set which contains a linearly dependent set is linearly dependent.
      ii. (1%) Any subset of a linearly independent set is linearly independent.
      iii. (1%) Any set which contains the $\mathbf{0}$ vector is linearly dependent.
      iv. (1%) A set S of vectors is linearly independent if and only if each finite subset of S is linearly independent.

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v. (1%) Any set which contains a linearly independent set is linearly independent.

7. **(20%)** Theorem 2. Let V and W be vector spaces and let T be a linear transformation from V into W. Suppose that V is n-dimensional with a basis $\{\boldsymbol{\alpha}_1, \ldots, \boldsymbol{\alpha}_n\}$. Let $\boldsymbol{\alpha}_1, \ldots, \boldsymbol{\alpha}_k$, $k < n$, be a basis of N, the null space of T. Then $\{T\boldsymbol{\alpha}_{k+1}, \ldots, T\boldsymbol{\alpha}_n\}$ is a basis that span the range of T.
   (a) (5%) State the definition of the linear transformation T.
   (b) (5%) State the definitions of the null space and the range of the linear transformation T.
   (c) (10%) Prove Theorem 2 from showing that the vectors $T\boldsymbol{\alpha}_{k+1}, \ldots, T\boldsymbol{\alpha}_n$ span the range of T and are linearly independent.

8. **(10%)** Let $\boldsymbol{A}$ and $\boldsymbol{B}$ be $n \times n$ matrices. Suppose that $\boldsymbol{I} - \boldsymbol{BA}$ with $\boldsymbol{I}$ an $n \times n$ identity matrix, is invertible. Prove that

$$(\boldsymbol{I} - \boldsymbol{BA})^{-1} = \boldsymbol{I} + \boldsymbol{B}(\boldsymbol{I} - \boldsymbol{AB})^{-1}\boldsymbol{A}.$$

備 註	一、作答於試題上者，不予計分。 二、試題請隨卷繳交。