1. (10 points) Let $X_1, X_2, \ldots, X_5$ be independent identically distributed (i.i.d.) from normal distribution $N(0, \sigma^2)$. Find the constant $c$ so that

$$c \cdot \frac{X_1 - X_2}{\sqrt{X_3^2 + X_4^2 + X_5^2}}$$

has a Student's $t$-distribution. How many degrees of freedom are associated with this random variable?

2. (10 points) Let $X_1, X_2, \ldots, X_n$ be a random sample from a normal distribution $N(0, \sigma^2)$, where the population variance $\sigma^2 > 0$ is unknown. Define

$$Q = \left(\frac{\overline{X}}{\sigma/\sqrt{n}}\right)^2 + \frac{(n-1)S^2}{\sigma^2},$$

where $\overline{X}$ and $S^2$ are the sample mean and sample variance, respectively.

(a) (5 points) Find the pdf of $Q$. Can it be used as a pivot?

(b) (5 points) Use $Q$ to find a $(1-\alpha) \times 100\%$ confidence interval for $\sigma^2$.

3. (15 points) Let $Y_1, Y_2, \ldots, Y_n$ be i.i.d. from one parameter exponential family with pdf or pmf $f(y|\theta)$ with the complete sufficient statistic $T(\mathbf{Y}) = \sum_{i=1}^n t(Y_i)$ where $t(Y_i) \sim \theta X$ and $X$ is a known distribution with known mean $E(X)$ and known variance $V(X)$. Let $W_n = cT(\mathbf{Y})$ be an estimator of $\theta$ where $c$ is a constant.

(a) (5 points) Find the mean square error (MSE) of $W_n$ as a function of $c$.

(b) (5 points) Find the value of $c$ that minimizes the MSE.

(c) (5 points) Find the uniformly minimum variance unbiased estimator (UMVUE) of $\theta$.

4. (15 points) Let $X_1, X_2, \ldots, X_n$ be a random sample from a discrete random variables having probability mass function

$$f(x|\theta) = \frac{2x}{\theta(\theta+1)}, \quad x \in \{1, 2, 3, \ldots, \theta\};$$

for some integer $\theta \geq 1$.

(a) (5 points) Find the maximum likelihood estimator of $\theta$.

(b) (5 points) Find the method of moments estimator of $\theta$.

(c) (5 points) Find the asymptotic distribution for your estimator in part (b) by the Central Limit Theorem.

Facts: $\sum_{i=1}^n i^2 = \frac{n(n+1)(2n+1)}{6}$ and $\sum_{i=1}^n i^3 = \frac{n^2(n+1)^2}{4}$

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5. (10 points) Let $Y_1, Y_2, \ldots, Y_n$ be a random sample from $N(\theta, \theta)$, where $\theta$ is an unknown parameter. Construct a confidence interval for $\theta$.

6. (10 points) Let $Y_1, Y_2, \ldots, Y_n$ be a random sample from a population with the following density function:

$$f_\theta(y) = \frac{a}{\theta}\left(\frac{y}{\theta}\right)^{a-1} I_{(0,\theta)}(y),$$

where $a \geq 1$ is known and $\theta > 0$ is unknown. Construct a confidence interval for $\theta$.

7. (10 points) Let $(Y_{i1}, Y_{i2})$, where $i = 1, 2, \ldots, n$, be a random sample from a bivariate normal distribution with unknown mean vector and covariance matrix. Find a likelihood ratio test for evaluating $H_0 : \rho = 0$ and $H_1 : \rho \neq 0$, where $\rho$ is the correlation coefficient.

8. (10 points) Let $Y_1, Y_2, \ldots, Y_n$ be a random sample from $N(\mu, \sigma^2)$, where $\mu$ and $\sigma^2$ are unknown parameters. Show that the power function of the one-sample $t$-test depends on a non-central $t$-distribution and it is an increasing function of $(\mu - \mu_0)/\sigma$ for testing $H_0 : \mu \leq \mu_0$ and $H_1 : \mu > \mu_0$.

9. (10 points) Let $\boldsymbol{Y}_1$ and $\boldsymbol{Y}_2$ denote $n_1 \times 1$ and $n_2 \times 1$ independent multivariate normal random vectors, denoted by

$$\boldsymbol{Y}_1 \sim N\left(\boldsymbol{X}_1\boldsymbol{\beta}_1, \sigma_1^2\boldsymbol{I}_{n_1}\right) \text{ and } \boldsymbol{Y}_2 \sim N\left(\boldsymbol{X}_2\boldsymbol{\beta}_2, \sigma_2^2\boldsymbol{I}_{n_2}\right),$$

respectively, where $\boldsymbol{X}_1$ is an $n_1 \times p$ matrix, $\boldsymbol{X}_2$ is an $n_2 \times p$ matrix, and $rank(\boldsymbol{X}_1) = rank(\boldsymbol{X}_2) = p$. Assume that $\sigma_2^2 = \Delta \times \sigma_1^2$ and $\Delta$ is a known constant. Find a test to evaluate $H_0 : \boldsymbol{\beta}_1 = \boldsymbol{\beta}_2$ and $H_1 : \boldsymbol{\beta}_1 \neq \boldsymbol{\beta}_2$.

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