1. Let $X_i \sim N(\mu_i, \sigma_i^2)$ for $i = 1, \ldots, 5$. X_i are uncorrelated. Use X_i to construct a statistic with the indicated distributions.
(i) χ_3^2, (3 points)
(ii) t-distribution with $df = 3$, (3 points)
(iii) F-distribution with $df = (2, 3)$. (4 points)

2. Let $X \sim beta(\alpha, \beta)$, where $\alpha > 0$ and $\beta > 0$. Given the pdf of X

$$f(x) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1},\ 0 < x < 1,$$

(i) Find $\mathbb{E}X^n$ and $\mathrm{var}(X)$. (10 points)
(ii) Use (i) to obtain the variance of $U(0, 1)$. (5 points)

3. Let $X_1, \ldots, X_n$ be i.i.d. random variables from $U(\alpha, \beta)$, where $\alpha < \beta$. What are the MLEs of α and β? (10 points)

4. Let $X_1, \ldots, X_n$ be i.i.d. random variables with cdf $F(\cdot)$, and let the statistic

$$F_n(x) = \frac{1}{n}\sum_{i=1}^{n} I(X_i \le x),$$

where $I(\cdot)$ is an indicator function.
(i) Show $\mathbb{E}[F_n(x)] = F(x)$. (5 points)
(ii) Find $\mathrm{var}[F_n(x)]$. (5 points)
(iii) Is $F_n(x)$ an UMVUE of $F(x)$? Why? (5 points)

5. Let $X_1, \ldots, X_n$ be i.i.d. random variables from $N(\mu, \sigma^2)$. Assume σ is known. Use the likelihood ratio test to $H_0 : \mu = \mu_0$ v.s. $H_1 : \mu \neq \mu_0$ at the level of significance α.
(15 points)

6. Let Y_n be a sequence of random variables satisfying $\sqrt{n}(Y_n - \mu) \to N(0, \sigma^2)$ in distribution. For a given function g and a specific value of μ, suppose $g'(\mu) = 0$ and $g''(\mu)$ exists and is not zero. Show

$$n[g(Y_n) - g(\mu)] \to \sigma^2 \frac{g''(\mu)}{2} \chi_1^2 \text{ in distribution.}$$

(15 points)

7. Let $Y_i = bx_i + \epsilon_i$ for $i = 1, \ldots, n$, where $x_1, \ldots, x_n$ are fixed constants, $\epsilon_1, \ldots, \epsilon_n$ are *i.i.d.* $N(0, \sigma^2)$ and σ^2 is unknown.
(i) Find the MLE of b, denoted as $\hat{b}$. Show that $\hat{b}$ is unbiased, and find $\mathrm{var}(\hat{b})$. (10 points)
(ii) Show that $\hat{b}_1 = \sum_{i=1}^n Y_i / \sum_{i=1}^n x_i$ is an unbiased estimator of b, and calculate $\mathrm{var}(\hat{b}_1)$.
(5 points)
(iii) Compare $\hat{b}$ and $\hat{b}_1$, which one is better? (5 points)