國立中正大學
112學年度碩士班招生考試
試題
[第1節]

科目名稱	數學
系所組別	資訊工程學系-甲組、乙組

—作答注意事項—

※作答前請先核對「試題」、「試卷」與「准考證」之系所組別、科目名稱是否相符。

1. 預備鈴響時即可入場，但至考試開始鈴響前，不得翻閱試題，並不得書寫、畫記、作答。
2. 考試開始鈴響時，即可開始作答；考試結束鈴響畢，應即停止作答。
3. 入場後於考試開始40分鐘內不得離場。
4. 全部答題均須在試卷（答案卷）作答區內完成。
5. 試卷作答限用藍色或黑色筆（含鉛筆）書寫。
6. 試題須隨試卷繳還。

1. (20 points) Multiple choices questions, gain two points for each correct answer.

   (a) Consider the homogeneous linear system $A\mathbf{x} = \mathbf{0}$, where $A=\begin{bmatrix} 1 & 4 & 0 & 1 \\ 1 & 4 & -3 & -3 \\ 0 & 0 & 3 & 4 \end{bmatrix}$, which one of the following statements is correct?

   (1). $A\mathbf{x} = \mathbf{0}$ has no solution.
   (2). Dimension of null space of A is 2.
   (3). Dimension of null space of A is 3.
   (4). For any vector $\mathbf{b} \in R^3$, $A\mathbf{x} = \mathbf{b}$ has at least one solution.

   (b) Which one of the following subset of all 2×2 real matrices is a vector subspace.

   (1). All 2×2 anti-symmetric matrices that is $A^T = -A$.
   (2). All 2×2 invertible matrices.
   (3). All 2×2 singular matrices.
   (4). All 2×2 matrices that satisfy the property $A^2 = \mathbf{0}$.

   (c) Let A be a 4×4 invertible matrix. Which one of the following statement is incorrect?

   (1). $A = E_1 E_2 \cdots E_k$ where each E_i is a 4×4 elementary matrix.
   (2). $det(A) = 0$.
   (3). $A\mathbf{x} = \mathbf{b}$ has a unique solution.
   (4). $(A^{-1})^T = (A^T)^{-1}$.

   (d) Let A be a 3×3 matrix with eigenvalues $-1, 2, 4$. Which one of the following statemnts is incorrect?

   (1). A is invertible.
   (2). A is diagonalizable.
   (3). Trace of A : $tr(A) = 7$
   (4). $det(A) = -8$

   (e) Which one of the following statements is not a basis for the vector space of all symmetric 2×2 matrices.

   (1). A= $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$.
   (2). A= $\begin{bmatrix} 3 & 3 \\ 3 & 3 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$.
   (3). A= $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} -2 & -2 \\ -2 & 1 \end{bmatrix}$.
   (4). A= $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} -1 & 2 \\ 2 & -1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

   (f) Let A be a 4×7 matrix. Which one of the following statements is incorrect?

   (1). The maximum possible value of $rank(A)$ is 4.
   (2). If $rank(A) = 3$, then dimension of column space of A is 4.
   (3). If $rank(A) = 4$, then $A\mathbf{x} = \mathbf{0}$ has infinite number of solutions.
   (4). $Rank(A) + Nullity(A) = 7$

(g) Let L be a lower triangular matrix. Which one of the following is incorrect?

(1). If L is invertible, then L^{-1} is an upper triangular matrix.

(2). $det(L)$ is the product of diagonal elements in L.

(3). If L is a square matrix then L^2 is a lower triangular matrix.

(4). L^T is an upper triangular matrix.

(h) Let A be a 3×3 matrix with eigenvalues : $2, 3, 4$ which one of the following statements is incorrect?

(1). A^{-1} has eigenvalues $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}$.

(2). A^2 has eigenvalues $4, 9, 16$.

(3). A^T has eigenvalues $2, 3, 4$.

(4). $A + 5I$ has eigenvalues $2, 3, 4$.

(i) Let V be the real vector space of continuous function over $[-1, 1]$ with the inner product $< f, g >= \int_{-1}^{1} f(x)g(x)dx$. Which one of the following statements is correct.

(1). $1, e^x$ is orthogonal.

(2). $1, x^2$ is orthogonal.

(3). e^x, e^{-x} is orthogonal.

(4). x, x^2 is orthogonal.

(j) Which one of the following statements is incorrect?

(1). If $\mathbf{b}$ is in column space of A, then the least square solution to $A\mathbf{x} = \mathbf{b}$ is also a solution to $A\mathbf{x} = \mathbf{b}$.

(2). The normal system $A^T A\mathbf{x} = A^T\mathbf{b}$ is always consistent.

(3). The least square solution to a system $A\mathbf{x} = \mathbf{b}$ is always unique.

(4). The least square solution to $A\mathbf{x} = \mathbf{b}$ is the projection of $\mathbf{b}$ in the column space of A.

2. (10 points)

(a) (5 points) Briefly explain spectral decomposition about matrix property, eigenvalues and diagonalization. Then give the definition of spectral decomposition.

(b) (5 points) Briefly explain singular value decomposition about matrix property, singular values, diagonalization and orthonormal matrices. Then give the definition of singular value decomposition.

3. (10 points) The matrix $A = \begin{bmatrix} 1 & 2 & 1 & -7 \\ 2 & 4 & 1 & -5 \\ 1 & 2 & 2 & -16 \end{bmatrix}$ is converted to row-reduced echelon form by Gaussian elimination, resulting the following matrix $R = \begin{bmatrix} 1 & 2 & 0 & 2 \\ 0 & 0 & 1 & 9 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. Give an orthonormal basis for the row space of A.

4. (10 points) Consider the basis $S=\{\mathbf{v_1},\mathbf{v_2}\}$ for R^2, where $\mathbf{v_1}=(1,1)^T$ and $\mathbf{v_2}=(1,0)^T$, and let $T: R^2 \rightarrow R^2$ be the linear operator for which : $T(\mathbf{v_1})=(1,-2)^T$ and $T(\mathbf{v_1}) = (-4,1)^T$.
   (a) (4 points) Compute $T(5,-3)$.
   (b) (6 points) Find a formula for $T(\mathbf{x_1},\mathbf{x_2})$.

5. (18 points) Let $Q(x,y)$ be the statement $x+y=x-y$. Determine the truth value of each of these statements if the universe of discourse for both variables consists of all integers.
   (a) $Q(1,1)$
   (b) $Q(2,0)$
   (c) $\forall y Q(1,y)$
   (d) $\exists x Q(x,2)$
   (e) $\exists x \exists y Q(x,y)$
   (f) $\forall x \exists y Q(x,y)$
   (g) $\exists y \forall x Q(x,y)$
   (h) $\forall y \exists x Q(x,y)$
   (i) $\forall x \forall y Q(x,y)$

6. (10 points) How many numbers must be selected from the first 12 positive integers to guarantee that at least three pairs of these numbers add up to 13?

7. (10 points) A string that contains only 0s and 1s is called a binary string.
   (a) (5 points) Find a recurrence relation for the number of binary strings of length n that contain a pair of consecutive 0s.
   (b) (2 points) What are the initial conditions?
   (c) (3 points) How many binary strings of length 7 do not contain two consecutive 0s?

8. (12 points) A simple graph is called regular if every vertex of this graph has the same degree. The complementary graph $\overline{G}$ of a simple graph G has the same vertices as G. Two vertices are adjacent in $\overline{G}$ if and only if they are not adjacent in G.
   (a) (4 points) How many vertices does a regular graph of degree 6 with 36 edges have?
   (b) (4 points) If G is a simple graph with 50 edges and $\overline{G}$ has 16 edges, how many vertices does G have?
   (c) (4 points) If the simple graph G has x vertices and y edges, how many edges does $\overline{G}$ have?