國立交通大學 101 學年度碩士班考試入學試題

科目：線性代數(4032)(4042)　　考試日期：101 年 2 月 17 日 第 2 節

系所班別：應用數學系　組別：應數系甲組，乙組　　第 1 頁，共 1 頁

【不可使用計算機】＊作答前請先核對試題、答案卷(試卷)與准考證之所組別與考科是否相符！！

1. Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be defined by $T(a_1, a_2) = (a_1 - a_2, a_1, 2a_1 + a_2)$. Let $\beta = \{(1,0),(0,1)\}$ be the standard ordered basis for $\mathbb{R}^2$ and $\gamma = \{(1,1,0),(0,1,1),(2,2,3)\}$.
   (a) (6%) Prove that γ is a basis for $\mathbb{R}^3$.
   (b) (6%) Compute $[T]_\beta^\gamma$, the matrix representation of T in the ordered bases β and γ.

2. (12%) Let V and W be vector spaces over a field F. Let $T : V \to W$ be a function satisfying $T(x+y) = T(x) + T(y)$. If F is the set of rational numbers, then prove or disprove that T is a linear transformation from V to W.

3. Let V and W be finite dimensional vector spaces and let $T : V \to W$ be linear and one-to-one.
   (a) (7%) Let $\alpha = \{v_1, v_2, \ldots, v_n\}$ be a basis for V. Prove that $\{T(v_i)\}_{i=1}^n$ is linearly independent.
   (b) (7%) Let the dimension of V be denoted by $\dim(V)$. Prove that $\dim(V) = \dim(W)$.

4. (12%) Let A be an $m \times n$ matrix with rank k. Suppose that B is an $m \times m$ matrix such that $BA = O$, where O is the zero matrix. Prove that $\text{rank}(B) \le m - k$.

5. Let V be a vector space and let $T : V \to V$ be linear. If $B = \begin{bmatrix} O & A \\ A & O \end{bmatrix}$ is the matrix representation of T, where O is the $n \times n$ zero matrix and A is an $n \times n$ matrix whose entries are all equal to one, then
   (a) (5%) find the range of T;
   (b) (5%) find the nullity of T;
   (c) (5%) find all eigenvalues of T.

6. (15%) Let $A \in \mathbb{R}^{n\times n}$ be nonsingular, $I \in \mathbb{R}^{k\times k}$ be an identity matrix, and $U, V \in \mathbb{R}^{n\times k}$, where $k \le n$. Prove that if $I + V^\top A^{-1} U$ is invertible, then
$$(A + UV^\top)^{-1} = A^{-1} - A^{-1}U(I + V^\top A^{-1} U)^{-1} V^\top A^{-1}.$$

7. Let A be an $n\times n$ anti-diagonal matrix (i.e., all the entries are zero except those on the diagonal going from the lower left corner to the upper right corner). If all anti-diagonal entries of A are one, then
   (a) (5%) find the trace of A;
   (b) (5%) compute A^2;
   (c) (5%) find the minimal polynomial of A;
   (d) (5%) find the Jordan canonical form of A.