國立成功大學

112學年度碩士班招生考試試題

編　號：35

系　所：數學系應用數學

科　目：線性代數

日　期：0207

節　次：第 1 節

備　註：不可使用計算機

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※ 考生請注意：本試題不可使用計算機。 請於答案卷(卡)作答，於本試題紙上作答者，不予計分。

On this exam V is a finite dimensional vector space over the field $\mathbb{F}$, where either $\mathbb{F} = \mathbb{R}$ or $\mathbb{F} = \mathbb{C}$. Also, $\mathbb{F}^n$ denotes the vector space of n-tuples with n entries from $\mathbb{F}$, as usual. The vector space of all linear maps from V to itself is denoted $\mathcal{L}(V)$. T^* is the adjoint of the operator T and $\bar{\lambda}$ is the complex conjugate of the scalar λ.

1. (a) (12 points) Let

$$A = \begin{bmatrix} 2 & -1 & 3 & -1 & 1 & -5 \\ -1 & 4 & 2 & 11 & -3 & 10 \\ 1 & 1 & 3 & 4 & 2 & -7 \end{bmatrix}$$

Find bases for the null and column spaces of A.

(b) Suppose A is a nilpotent matrix.

i. (7 points) Prove that $I - A$ is invertible. Find $(I - A)^{-1}$.

ii. (3 points) Prove that $I + A$ is invertible.

2. (10 points) Define a real vector space $V = \{p(x) \mid p(x) = ax^2 + bx + c, a, b, c \in \mathbb{R}\}$, with inner product $\langle p, q\rangle = \int_0^1 p(x)q(x)\,dx$. Find an orthonormal basis for V.

3. (a) (12 points) Suppose $T \in \mathcal{L}(V)$. Prove that the zeros of the minimal polynomial of T are precisely the eigenvalues of T.

(b) Suppose $T \in \mathcal{L}(\mathbb{F}^4)$ is defined by $T(w_1, w_2, w_3, w_4) = (0, w_2 + w_4, w_3, w_4)$.

i. (8 points) Determine the minimal polynomial of T.

ii. (4 points) Determine the characteristic polynomial of T.

4. (10 points) Suppose $T \in \mathcal{L}(V)$ is normal and $v \in V$ is an eigenvector of T with eigenvalue λ. Prove that v is also an eigenvector of the adjoint T^* with eigenvalue $\bar{\lambda}$.

5. (14 points) Suppose $P \in \mathcal{L}(V)$ is such that $P^2 = P$. Prove that P is an orthogonal projection if and only if P is self-adjoint.

6. (a) (6 points) Suppose V is an inner product space and $T \in \mathcal{L}(V)$. Prove that if $e_1, \ldots, e_n$ is an orthonormal basis of V, then

$$\text{trace}\,(T^*T) = \|Te_1\|^2 + \cdots + \|Te_n\|^2.$$

Conclude that the right side of the equation above is independent of which orthonormal basis $e_1, \ldots, e_n$ is chosen for V.

(b) (14 points) Suppose V is a complex inner product space and $T \in \mathcal{L}(V)$. Let $\lambda_1, \ldots, \lambda_n$ be the eigenvalues of T, repeated according to multiplicity. Suppose

$$\begin{bmatrix} a_{1,1} & \cdots & a_{1,n} \\ \vdots & & \vdots \\ a_{n,1} & \cdots & a_{n,n} \end{bmatrix}$$

is the matrix of T with respect to some orthonormal basis of V. Prove that

$$|\lambda_1|^2 + \cdots + |\lambda_n|^2 \leq \sum_{k=1}^{n}\sum_{j=1}^{n} |a_{j,k}|^2.$$