國立交通大學 101 學年度碩士班考試入學試題

科目：高等微積分(4031)　　考試日期:101 年 2 月 17 日 第 1 節

系所班別：應用數學系　　組別：應數系甲組　　

【不可使用計算機】*作答前請先核對試題、答案卷(試卷)與准考證之所組別與考科是否相符！！

1. (12 points) Let K be a compact set on a metric space (M, d_M). Show that every closed subset of K is compact.

2. (12 points)

   (a) (6 points) Suppose $f : \mathbb{R}^m \longrightarrow \mathbb{R}^n$ is a continuous function and the set $A \subseteq \mathbb{R}^m$ is bounded. Show that $f(A)$ is bounded.

   (b) (6 points) Let (M, d_M) and (N, d_N) be metric spaces. Suppose $f : M \longrightarrow N$ is a continuous function and $A \subseteq M$ is bounded. Is $f(A)$ bounded? Justify your answer.

3. (14 points) Consider the function $f_n(x) = \displaystyle\sum_{m=1}^{n} \frac{1}{2^m} \sin(mx)$ for $x \in \mathbb{R}$.

   (a) (8 points) Is the function $f(x) := \lim_{n\to\infty} f_n(x)$ continuous? Justify your answer.

   (b) (6 points) Find the value of $\displaystyle\int_0^{2\pi} f(t)\,dt$.

4. (12 points) Using the fact

$$\sin\frac{\pi}{n}\,\sin\frac{2\,\pi}{n}\cdots\sin\frac{(n-1)\,\pi}{n} = \frac{n}{2^{n-1}}$$

   to evaluate the improper integral $\displaystyle\int_0^{\pi} \log(\sin x)\,dx$.

5. (18 points) Let φ be a continuous function defined on the unit circle $\{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$ such that $\varphi(1, 0) = \varphi(0, 1) = 0$ and $\varphi(-x, -y) = -\varphi(x, y)$. Define $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ by

$$f(x, y) = \begin{cases} \sqrt{x^2+y^2}\cdot\varphi\left(\frac{x}{\sqrt{x^2+y^2}}, \frac{y}{\sqrt{x^2+y^2}}\right) & \text{if } (x, y) \neq (0, 0), \\ 0 & \text{if } (x, y) = (0, 0). \end{cases}$$

   (a) (10 points) If $(x, y) \in \mathbb{R}^2$ is arbitrarily fixed, define $g(t) = f(tx, ty)$ for $t \in \mathbb{R}$. Is g differentiable? Justify your answer.

   (b) (8 points) Find a necessary and sufficient condition so that the f defined above is differentiable at $(0, 0)$. Justify your answer.

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6. (12 points) Let $F : \mathbb{R}^n \longrightarrow \mathbb{R}$. Suppose that all $\frac{\partial F}{\partial x_j}$ are continuous and that the equation $F(x_1, x_2, \ldots, x_n) = 0$ can be solved for x_j as a function of remaining $(n-1)$-variables, here $j = 1, 2, \ldots, n$. Compute

$$\frac{\partial x_n}{\partial x_1} \times \frac{\partial x_1}{\partial x_2} \times \frac{\partial x_2}{\partial x_3} \times \cdots \times \frac{\partial x_{n-1}}{\partial x_n}.$$

Justify your computation.

7. (10 points) Compute $\displaystyle\int_0^{\frac{\pi}{2}} \frac{\cos^n x}{\cos^n x + \sin^n x}\, dx$ for all positive integer n.

8. (10 points) Let $\overrightarrow{F} = (P, Q, R) : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ be a smooth vector field, and let Γ be a compact simple closed loop in $\mathbb{R}^3$. Prove or disprove that the surface integral

$$\iint_\Sigma \overrightarrow{F} \cdot \overrightarrow{n}_\Sigma \, dA$$

is independent of orientable Σ but only on Γ, if

$$\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} = 0.$$

Here $\overrightarrow{n}_\Sigma$ is the unit normal of Σ and the boundary of Σ is Γ.