

考試科目	線性代數	系所別	應用數學系	考試時間	2 月 3 日(五) 第四節
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Please show all your work.

- (10%) Let  $V$  and  $W$  be finite-dimensional vector spaces and  $T:V \rightarrow W$  be an isomorphism. Let  $V_0$  be a subspace of  $V$ . Prove that the image of  $V_0$  under  $T$ , denoted as  $T(V_0)$ , is a subspace of  $W$  and  $\dim(V_0) = \dim(T(V_0))$ .
- (20%) Let  $T$  be a linear operator on a finite-dimensional vector space  $V$ , and let  $W$  be a  $T$ -invariant subspace of  $V$ . Suppose that  $v_1, v_2, \dots, v_k$  are eigenvectors of  $T$  corresponding to distinct eigenvalues. Prove that if  $v_1 + v_2 + \dots + v_k$  is in  $W$ , then  $v_i \in W$  for all  $i$ .
- (20%) Let  $T$  be a linear operator on an  $n$ -dimensional vector space  $V$  such that  $T$  has  $n$  distinct eigenvalues. Prove that  $V$  is a  $T$ -cyclic subspace of itself.
- (15% for each sub-problem) Let  $V$  be a finite-dimensional inner product space over the complex field  $C$  with inner product  $\langle \cdot, \cdot \rangle$ . A linear operator  $U$  on  $V$  is called a partial isometry if there exists a subspace  $W$  of  $V$  such that  $\|U(x)\| = \|x\|$  for all  $x \in W$  and  $U(x) = 0$  for all  $x \in W^\perp$  where  $\|\cdot\|$  denotes the norm induced by the inner product and the set  $W^\perp$  denotes the orthogonal complement of  $W$ . Suppose that  $U$  is such an operator and  $\{v_1, v_2, \dots, v_k\}$  is an orthonormal basis for  $W$ . Prove that (a)  $\langle U(x), U(y) \rangle = \langle x, y \rangle$  for all  $x, y \in W$ .  
(b)  $\{U(v_1), U(v_2), \dots, U(v_k)\}$  is an orthonormal basis for  $R(U)$ , the range of  $U$ .

備

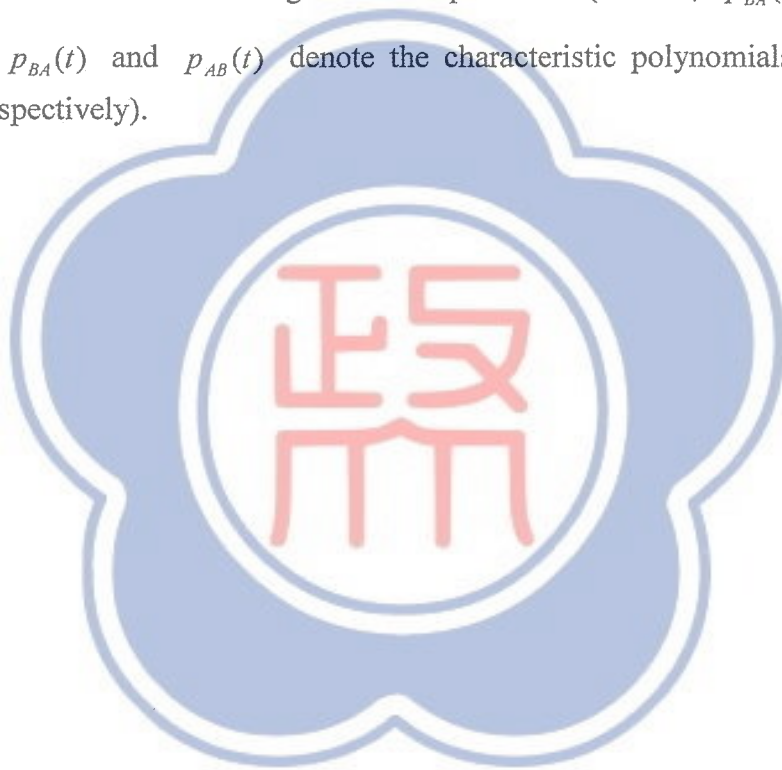
註

- 一、作答於試題上者，不予計分。
- 二、試題請隨卷繳交。

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5. (20%) Let  $M_{m,n}(R)$  denote the set of all  $m \times n$  matrices with real entries.

Suppose that  $A \in M_{m,n}(R)$  and  $B \in M_{n,m}(R)$  with  $m \leq n$ . Show that  $BA$  has the same eigenvalues as  $AB$  (including the corresponding multiplicities), together with an additional  $n - m$  eigenvalues equal to 0. (That is,  $p_{BA}(t) = t^{n-m} p_{AB}(t)$  where  $p_{BA}(t)$  and  $p_{AB}(t)$  denote the characteristic polynomials of  $BA$  and  $AB$ , respectively).



備

註

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