考試科目	線性代數	系所別	應用數學系	考試時間	2 月 3 日(五)第四節

Please show all your work.

1. **(10%)** Let V and W be finite-dimensional vector spaces and $T:V \to W$ be an isomorphism. Let V_0 be a subspace of V. Prove that the image of V_0 under T, denoted as $T(V_0)$, is a subspace of W and $\dim(V_0) = \dim(T(V_0))$.

2. **(20%)** Let T be a linear operator on a finite-dimensional vector space V, and let W be a T-invariant subspace of V. Suppose that $v_1, v_2, \ldots v_k$ are eigenvectors of T corresponding to distinct eigenvalues. Prove that if $v_1 + v_2 + \cdots + v_k$ is in W, then $v_i \in W$ for all i.

3. **(20%)** Let T be a linear operator on an n-dimensional vector space V such that T has n distinct eigenvalues. Prove that V is a T-cyclic subspace of itself.

4. **(15% for each sub-problem)** Let V be a finite-dimensional inner product space over the complex field C with inner product $\langle \bullet, \bullet \rangle$. A linear operator U on V is called a partial isometry if there exists a subspace W of V such that $\|U(x)\| = \|x\|$ for all $x \in W$ and $U(x) = 0$ for all $x \in W^\perp$ where $\|\bullet\|$ denotes the norm induced by the inner product and the set $W^\perp$ denotes the orthogonal complement of W. Suppose that U is such an operator and $\{v_1, v_2, \ldots v_k\}$ is an orthonormal basis for W. Prove that (a) $\langle U(x), U(y) \rangle = \langle x, y \rangle$ for all $x, y \in W$.

(b) $\{U(v_1), U(v_2), \ldots U(v_k)\}$ is an orthonormal basis for $R(U)$, the range of U.

備註	一、作答於試題上者，不予計分。 二、試題請隨卷繳交。

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5. **(20%)** Let $M_{m,n}(R)$ denote the set of all $m \times n$ matrices with real entries. Suppose that $A \in M_{m,n}(R)$ and $B \in M_{n,m}(R)$ with $m \leq n$. Show that BA has the same eigenvalues as AB (including the corresponding multiplicities), together with an additional $n-m$ eigenvalues equal to 0. (That is, $p_{BA}(t) = t^{n-m} p_{AB}(t)$ where $p_{BA}(t)$ and $p_{AB}(t)$ denote the characteristic polynomials of BA and AB, respectively).

備註	一、作答於試題上者，不予計分。 二、試題請隨卷繳交。