考 試 科 目	統計學B	所 別	金融學系 財務工程與金融科技組	考 試 時 間	2 月 10 日(四)第三節

1. (30%) Brownian Motion

Consider a Brownian motion process $\{B(t), t \geq 0\}, B(0) = 0$ and $B(t)$ is normal with mean 0 and variance t, and $\{B(t), t \geq 0\}$ has stationary and independent increments, where $B(t_1)$, $B(t_2) - B(t_1) \ldots B(t_n) - B(t_{n-1})$ for $t_1 < \cdots < t_n$ are independent and $B(t_k) - B(t_{k-1})$ is normal with mean 0 and variance $t_k - t_{k-1}$.

(1) Please give the joint probability density function of $B(t_1)$, $B(t_2)$...,$B(t_n)$ for $t_1 < \cdots < t_n$. (5%)

(2) Please find the covariance of $B(s)$ and $B(t)$, $Cov(B(s), B(t))$ for $s < t$. (5%)

(3) Please find the conditional distribution of $B(t)$ given $B(s) = C$, and calculate its conditional mean $\mathbb{E}(B(t)|B(s) = C)$ and conditional variance $Var(B(t)|B(s) = C)$ for $s < t$. (10%)

(4) Please find the conditional distribution of $B(t)$ given $B(s) = C$, and calculate its conditional mean $\mathbb{E}(B(t)|B(s) = C)$ and conditional variance $Var(B(t)|B(s) = C)$ for $t < s$. (10%)

2. (15%) Martingale

A martingale is a random process $X(T)$ satisfied $\mathbb{E}(|X(T)|) < \infty$ and $\mathbb{E}(X(T)|\mathcal{F}_t) = X(t)$ for $T > t$, where $\mathcal{F}_t$ is the filtration at time t.

(1) Please show that $\{Y(t), t \geq 0\}$ is a martingale where $Y(t) = B^2(t) - t$. (Hint: find $\mathbb{E}(Y(T)|\mathcal{F}_t, t < T)$). (5%)

(2) Suppose that we want to use Monte Carlo method to get the price of two-assets rainbow options, so we need to generate two Brownian motion processes $B_1(t)$ and $B_2(t)$ for $t \geq 0$, where they are correlated with correlation ρt, $\begin{pmatrix} B_1(t) \\ B_1(t) \end{pmatrix} \sim MN(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} t & \rho t \\ \rho t & t \end{pmatrix})$.Assume $Z_1(t)$ and $Z_2(t)$ are independent and identically distributed normal with mean 0 and variance t, $Z_i(t) \xrightarrow{i.i.d.} N(0, t), i = 1,2$. Please describe in detail how to convert two independent random variables $Z_1(t)$, $Z_2(t)$ into two correlated two random variables $B_1(t)$, $B_2(t)$. (10%)

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3. (55%) Ito's Lemma and Black-Scholes Pricing Formula

Let $B(t)$ be a Brownian motion and $X(t)$ be an Ito drift-diffusion process which satisfies the stochastic differential equation:

$$dX(t) = \mu(X(t),t)dt + \sigma(X(t),t)dB(t)$$

where $\mu(X(t),t)$ and $\sigma(X(t),t)$ are the drift term and diffusion term, respectively.

If $f(t,X(t))$ is twice-differentiable function, then the function f follows the process

$$df = \left(\frac{\partial f}{\partial t} + \frac{\partial f}{\partial X(t)}\mu(X(t),t) + \frac{1}{2}\frac{\partial^2 f}{\partial X^2(t)}\sigma^2(X(t),t)\right)dt + \frac{\partial f}{\partial X(t)}\sigma(X(t),t)dB(t).$$

(1). Do you think what is the advantage of Ito's Lemma for the finance and mathematics? (5%)

(2) Under physical probability measure $\mathcal{P}$, given the dynamic of stock price

$$dS_t = \mu S_t dt + \sigma S_t dB_t^{\mathcal{P}}$$

where dS_t denotes the stock price change at instantaneous time, μ presents the expected return of the stock at instantaneous time, σ means the volatility of the stock return, and $dB_t^{\mathcal{P}}$ is the change of the Brownian motion at instantaneous time under $\mathcal{P}$.

Please use Ito's lemma and show that the solution of the stochastic differential equation is

$$S_T = S_0 e^{(\mu - 0.5\sigma^2)T + \sigma\Delta B_T^{\mathcal{P}}}$$

(10%)

(3) According to the above results, we know that the stock price S_T is said to have a lognormal distribution. Please find the probability density function, the mean and the variance of the stock price S_T under $\mathcal{P}$. (10%)

(4). Please explain the difference between implied volatility and history volatility to calibration the variance of the return for the stock? (10%)?

(5) Please derive the Black-Scholes pricing formula (as you know method) at time 0 for European call option with the strike K, the maturity T, and the risk-free interest rate r. (10%)

(6) Please show that $\frac{\partial c_0}{\partial S_0} = N(d_1)$ and $\frac{\partial c_0}{\partial \sigma} = S_0\sqrt{T}\,n(d_1)$, where $n(\cdot)$ is the standard normal probability density function, $N(\cdot)$ is the standard normal cumulative density function, and $d_1 = \frac{\ln\left(\frac{S_0}{K}\right) + \left(r + \frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}$. (10%)

備註	一、作答於試題上者，不予計分。 三、試題請隨卷繳交。