1. (25%). Suppose a vector $u = \left(\frac{6\sqrt{2}}{13} \quad \frac{-3\sqrt{2}}{26} \quad \frac{-2\sqrt{2}}{13} \quad \frac{\sqrt{2}}{2}\right)$ and a 4 by 4 matrix $M = I - 2uu^T$.
   (a) (7%). Evaluate M^2 and M^TM.
   (b) (3%). Is M an orthogonal matrix?
   (c) (5%). One eigenvector of M is u itself. Find the corresponding eigenvalue.
   (d) (5%). If v is any vector perpendicular to u, show that v is an eigenvector of M.
   (e) (5%). Find the eigenvalue in part (d).

2. (30%). Consider the ordinary differential equation:
$$Y'' + 2Y' + 17Y = 60e^{-4t}\cos 5t$$
   Please answer the following questions:
   (a) (3%). What is the order of this O.D.E?
   (b) (7%). Find the homogeneous solution.
   (c) (16%). Find the particular solution.
   (d) (4%). Find the general solution.

3. (30%). Consider a torsionally vibrating shaft with a twist angle $\theta(x,t)$ that takes the form
$$a^2\frac{\partial^2\theta}{\partial x^2} = \frac{\partial^2\theta}{\partial t^2}$$
   where $0 < x < 1$ and $t > 0$. The shaft is fixed on one end $(x = 0)$ and free on the other end $(x = 1)$.
   (a) (5%). Determine the boundary conditions.
   (b) (15%). Find the eigenfunctions for this partial different equation.
   (c) (10%). Assume that $\theta(x,0) = x$ and $\frac{\partial\theta}{\partial t}\big|_{t=0} = 0$ for $0 < x < 1$, solve for $\theta(x,t)$.

4. (15%). The velocity field of a fluid at time t is given by
$$\boldsymbol{P}(x,y,z) = x^3\boldsymbol{i} + 2y^3\boldsymbol{j} + 5z\boldsymbol{k}$$
   Consider the fluid flux
$$\iint_{\partial F}\boldsymbol{P}\cdot\boldsymbol{n}\,dF$$
   through the surface ∂F of the closed cone $F = \{(x,y,z): z^2 = x^2 + y^2, 0 \le z \le 1\}$.
   (a) (5%). Use the divergence theorem to express the flux through ∂F in terms of an integral.
   (b) (10%). Evaluate the integral found in (a).

試題隨卷繳回